SIAM Conference on Imaging Science, 2018

Flexible Methodology for Image Segmentation

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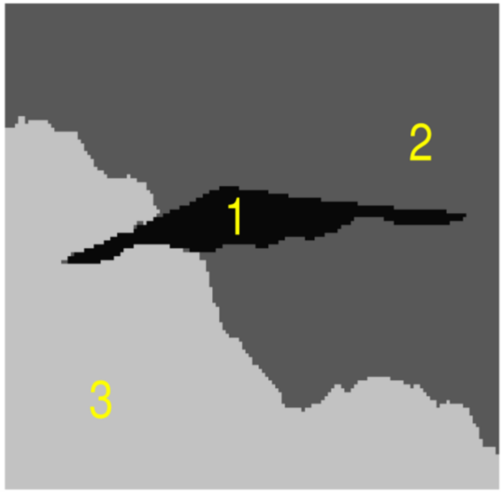
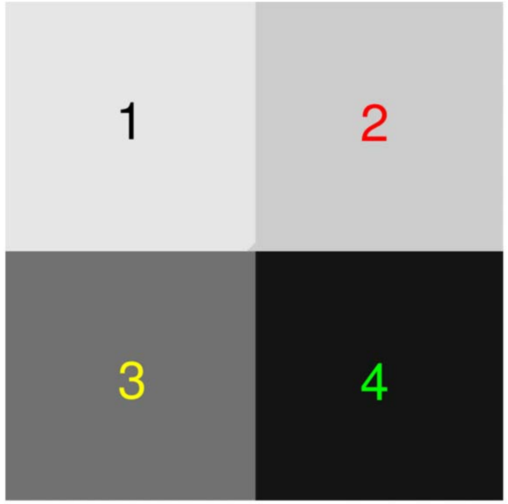
Aim: One Framework to Segment Different Images

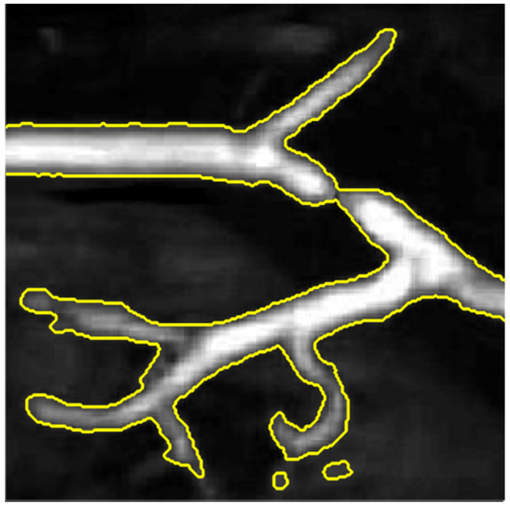

motion + Gaussian

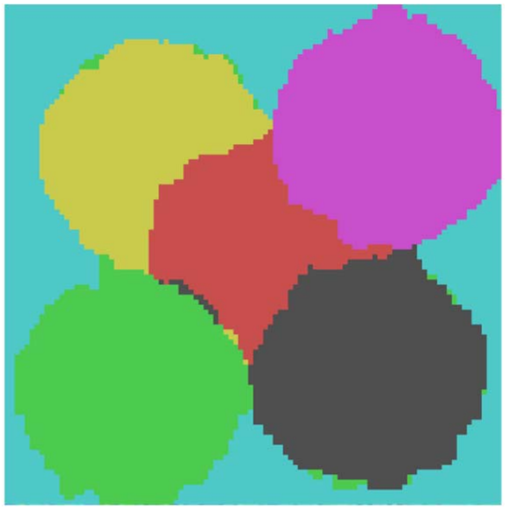


Gamma noise



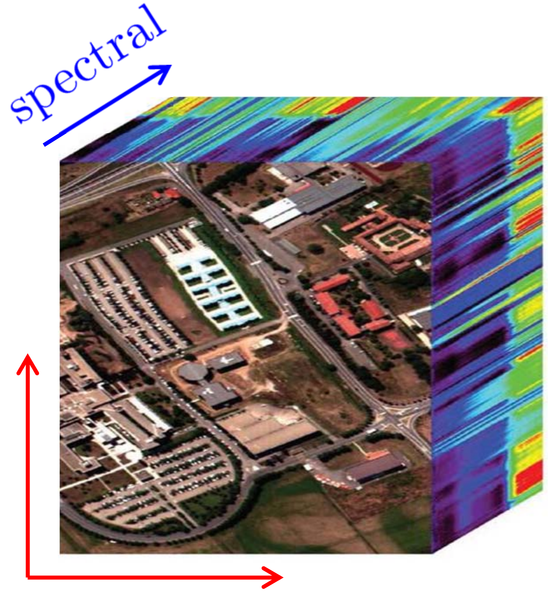
Poisson noise

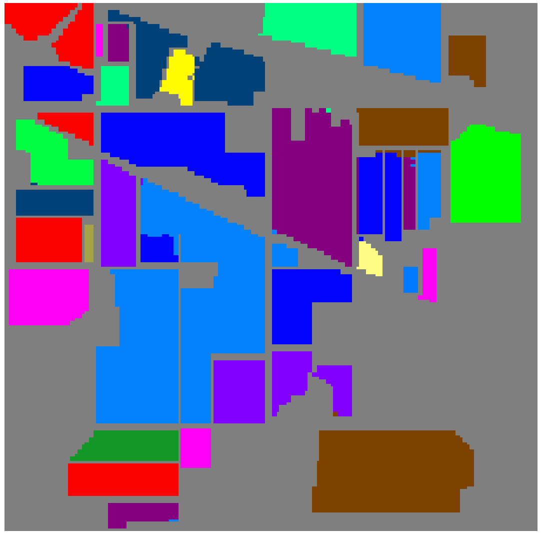

MRA image


color image
motion + Gaussian


60% pixel loss
+ Gaussian

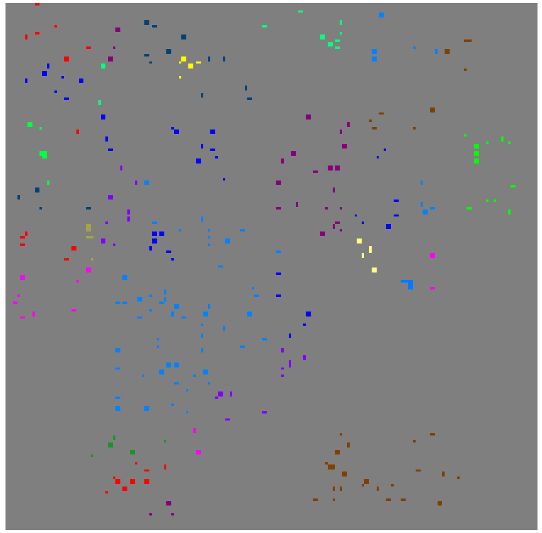
Hyper-spectral Image and Point Cloud Classification



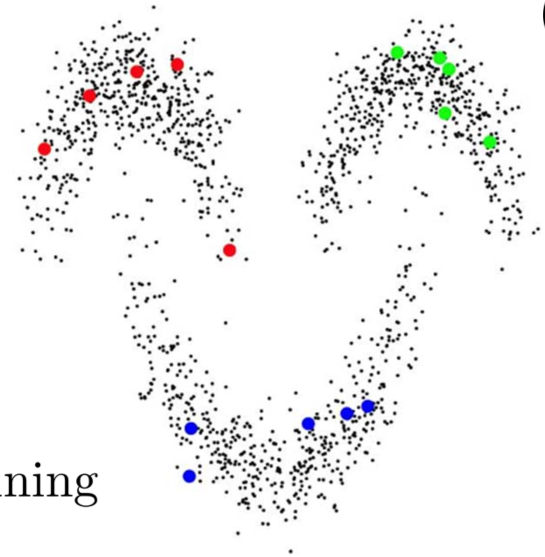


Ground-truth

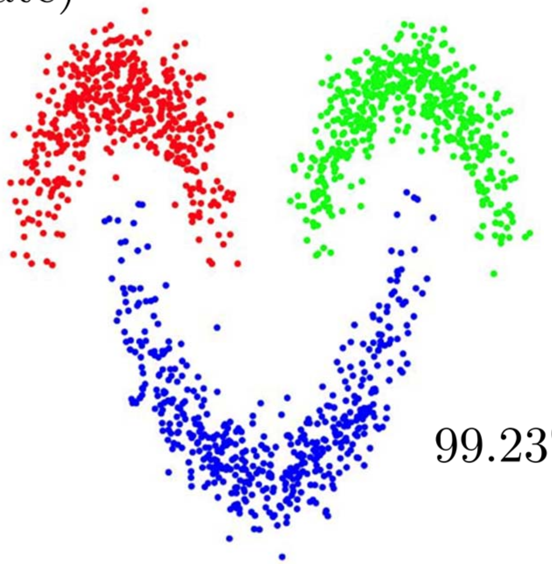
Our method
(98.88% accurate)



10% training pixels



10% training



99.23% accurate

Outline

1. Mumford-Shah Model

2. Two-stage Image Segmentation Method

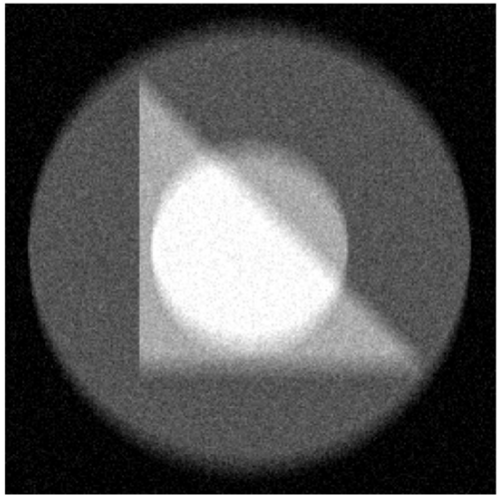
3. Three-stage Color Segmentation Model

4. Convex/Nonconvex Model

5. Hyperspectral Image and Point Cloud Classification

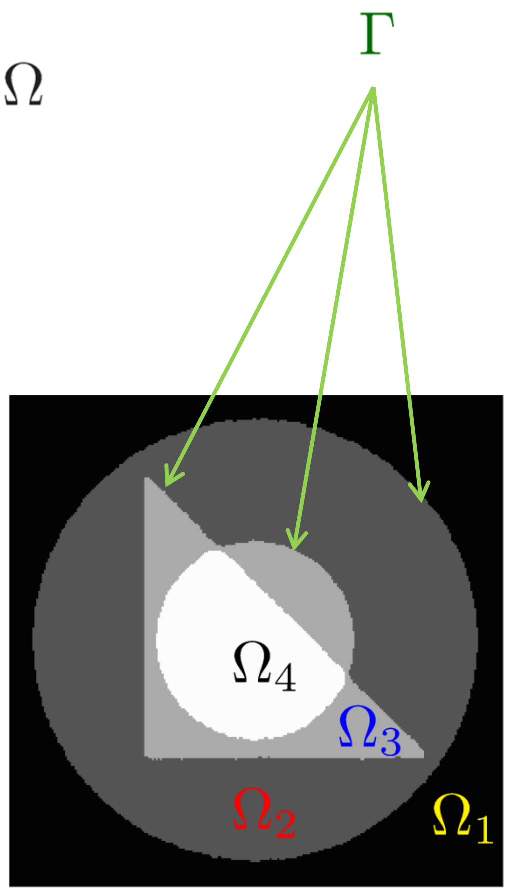
Segmentation: Problem Setting and Notation

Given a corrupted image f,

$f =$  $= \Omega$

want a K-phase segmentation $=$

find a piecewise constant approximation with K constant regions

$g =$ 

$K = 4$

$$\Omega \setminus \Gamma = \cup_i \Omega_i,$$
$$g = c_i \text{ in } \Omega_i,$$
$$i = 1, 2, 3, 4$$

Mumford-Shah Model (1989) [cited 5,700+ time]

f $\xrightarrow{\text{minimize}}$ Mumford-Shah Energy $E_{\mathrm{MS}}(g, \Gamma)$

$$E_{\mathrm{MS}}(g, \Gamma) = \frac{\lambda}{2}\int_{\Omega}(f-g)^2 dx + \frac{\mu}{2}\int_{\Omega \setminus \Gamma} |\nabla g|^2 dx + \mathrm{Length}(\Gamma)$$

Data fidelity: control g not far away from f

Regularization: impose smoothness of g on $\Omega \setminus \Gamma$

Regularization: require boundary Γ be short

Highly non-convex problem

Simplifying Mumford-Shah Model

Mumford-Shah Energy

$$E_{\mathrm{MS}}(g, \Gamma)$$

Simplify it:
$\nabla g \equiv 0$ on $\Omega \setminus \Gamma$

$$\Updownarrow$$

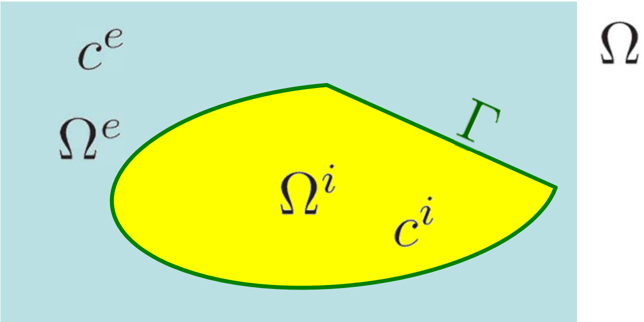
$$\frac{\lambda}{2} \int_\Omega (f - g)^2 dx + \cancelto{0}{\frac{\mu}{2} \int_{\Omega \setminus \Gamma} |\nabla g|^2 dx} + \mathrm{Length}(\Gamma)$$

Multiphase Chan-Vese Model (02)
(minimizer $\hat{g}$ is piecewise constant):

$$E_{\mathrm{MS}}(\{c_i\}, \Gamma) = \frac{\lambda}{2} \sum_{i=1}^{K} \int_{\Omega_i} (f - c_i)^2 + \mathrm{Length}(\Gamma)$$

2-phase Chan-Vese Model (2001) [cited 10,200+ times]

$$E_{\mathrm{CV}}(\Gamma, c^i, c^e) = \frac{\lambda}{2}\int_{\Omega^i}(f - c^i)^2 + \frac{\lambda}{2}\int_{\Omega^e}(f - c^e)^2 + \mathrm{Length}(\Gamma)$$



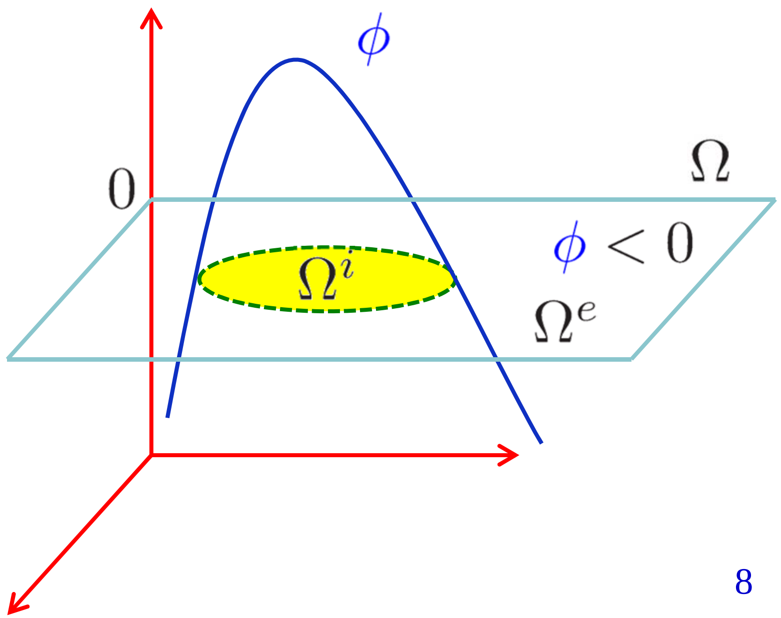
Minimize by Alternating Direction Method:

Step 1: Given Γ, minimize c^i and c^e.

1. c^i = average of f in Ω^i
2. c^e = average of f in Ω^e

Step 2: Given c^i and c^e, find Γ.

Express Γ as zero-contour of a level set function ϕ and minimize w.r.t. ϕ.



How to Solve the Level-set Function?

Minimizing ϕ in:

$$\frac{\lambda}{2}\int_{\Omega}(f-c^i)^2 H(\phi)+\frac{\lambda}{2}\int_{\Omega}(f-c^e)^2(1-H(\phi))+\int_{\Omega}\delta(\phi)|\nabla(\phi)|$$

where $H(\cdot)$ is the Heaviside function, and $\delta = H'$.

Euler-Lagrange equation for ϕ:

$$\delta(\phi)\left[\operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)+\lambda(f-c^i)^2-\lambda(f-c^e)^2\right]=0.$$

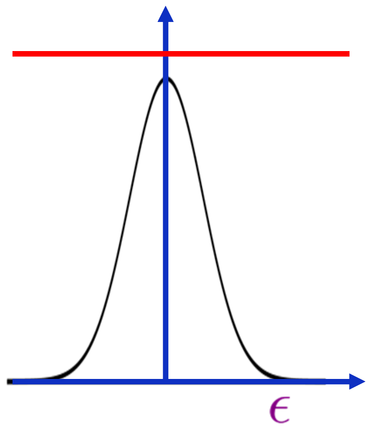
Change to parabolic and solve by time-marching:

$$\frac{\partial\phi}{\partial t}=\delta(\phi)\left[\operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)+\lambda(f-c^i)^2-\lambda(f-c^e)^2\right].$$

Convexified Chan-Vese Model

Smooth delta function $\delta(\cdot)$ by $\delta_\epsilon(\cdot)$:

$$\frac{\partial \phi}{\partial t} = \overset{1}{\cancel{\delta_\epsilon(\phi)}} \left[\operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|} \right) + \lambda (f - c^i)^2 - \lambda (f - c^e)^2 \right].$$



T. Chan, Esedoglu, and Nikolova (06):

$$\hat{\phi} = \arg \min_{0 \le \phi \le 1} \left\{ \int_\Omega \left[\frac{\lambda}{2} (f - c^i)^2 - \frac{\lambda}{2} (f - c^e)^2 \right] \phi(x) + \boxed{\int_\Omega |\nabla \phi|} \right\},$$

and $\Omega^i := \{x : \hat{\phi}(x) \ge \rho\}$ for a.e. $\rho \in [0, 1]$.

thresholding $\hat{\phi}$ by ρ to get Ω^i and hence Γ

Convexified Chan-Vese Model

$$E_{\mathrm{CV}}(\Gamma, c^i, c^e) = \frac{\lambda}{2}\int_{\Omega^i}(f - c^i)^2 + \frac{\lambda}{2}\int_{\Omega^e}(f - c^e)^2 + \mathrm{Length}(\Gamma)$$

Minimize by Alternating Direction Method:

$\boxed{\text{Step 1: Given } \Gamma, \text{ minimize } c^i \text{ and } c^e.}$

1. c^i = average of f in Ω^i
2. c^e = average of f in Ω^e

$\boxed{\text{Step 2: Given } c^i \text{ and } c^e, \text{ find } \Gamma.}$

Express Γ as zero-contour of a level set function ϕ and minimize w.r.t. ϕ.

ADM:
Many convex problems to solve

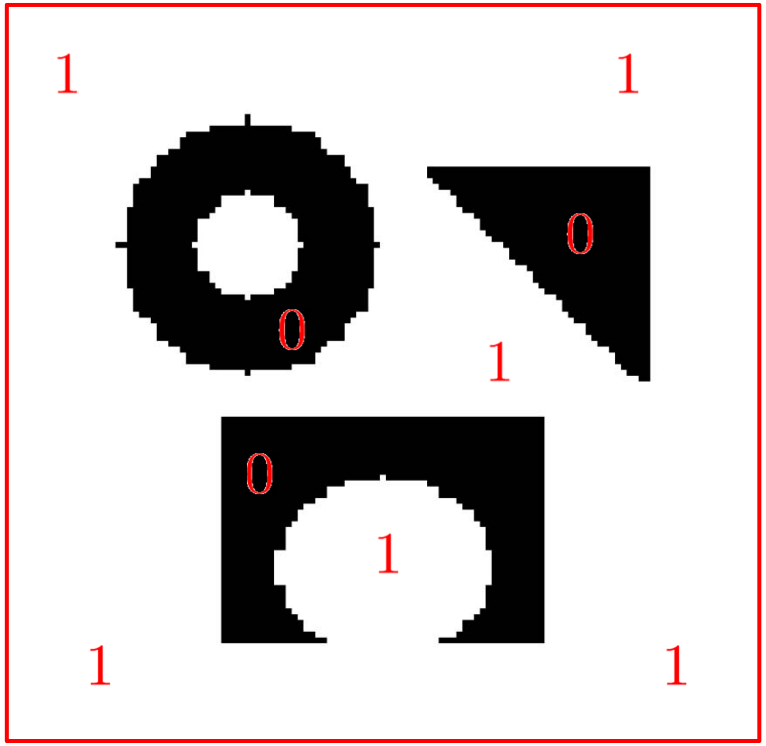
Ours:
Only 1 convex problem to solve

convexify ϕ
(T. Chan *et al.* (06))

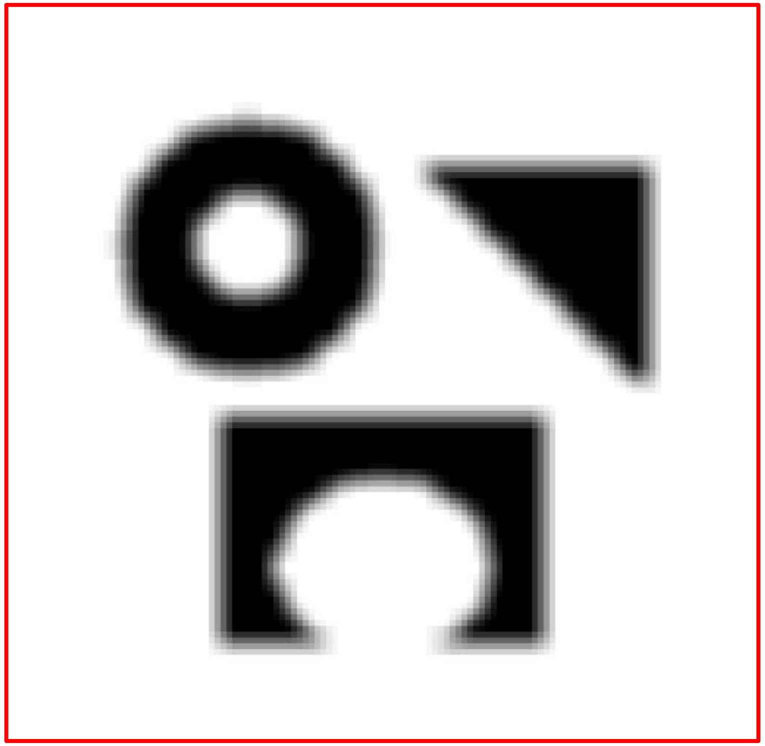
Outline

1. **Mumford-Shah Model**
2. **Two-stage Image Segmentation Method**
3. **Three-stage Color Segmentation Model**
4. **Convex/Nonconvex Model**
5. **Hyperspectral Image and Point Cloud Classification**

Our Motivation

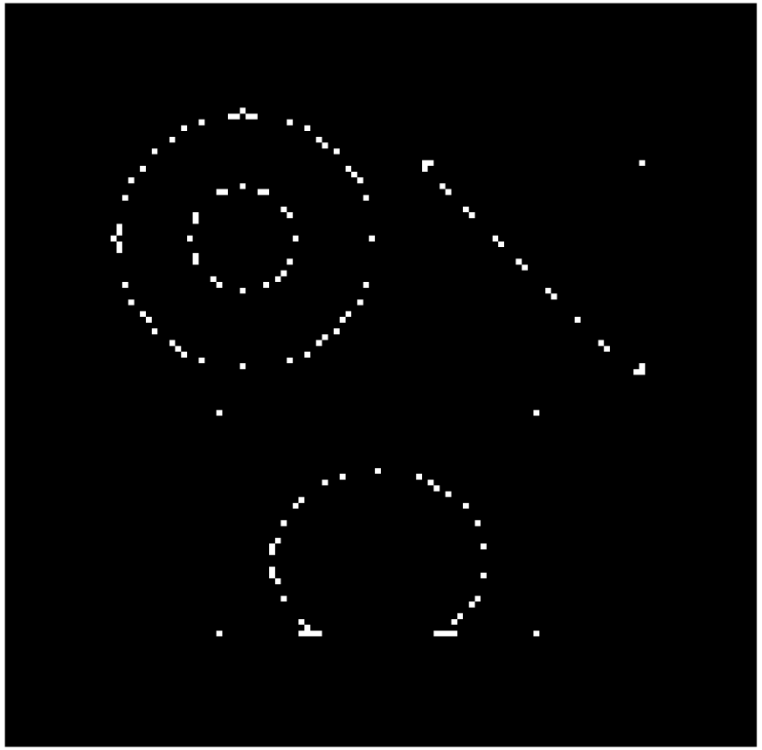


(a): True binary image



(b): Smooth image of (a)

Our Motivation



(d): Difference of (a) and (c)
(nonzero pixel values
only at the boundary)

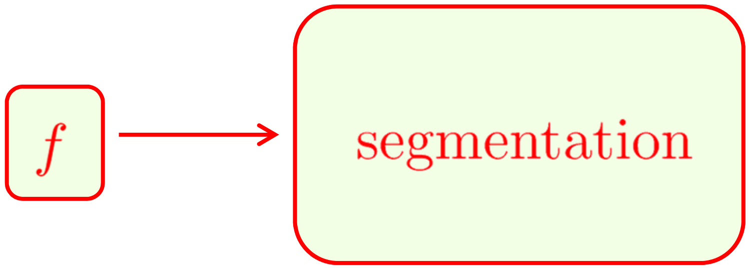
piecewise-constant
approximation

(d): Difference of (a) and (c)
(nonzero pixel values
only at the boundary)

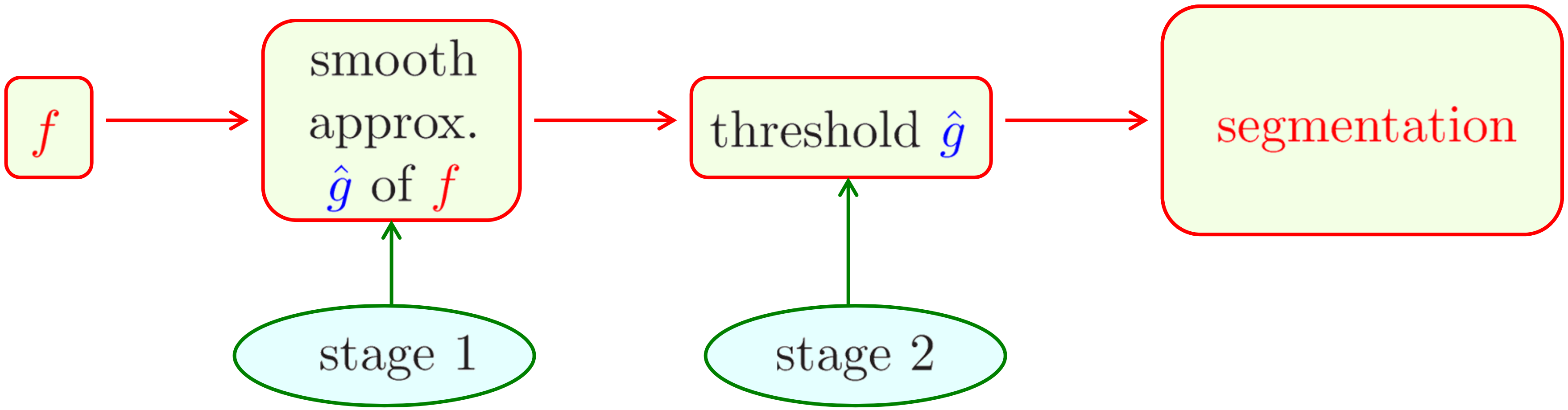
thresholding
smooth function

Aim in Segmentation

Chan-Vese: Get a piecewise constant approximation $\hat{g}$ of f



Our idea:



Stage One: Convex Variant of the M-S Model

$$\frac{\text{Mumford-Shah Energy}}{E_{\mathrm{MS}}(g, \Gamma)}$$

Restrict:
$g \in W^{1,2}(\Omega)$

$$\Downarrow$$

$$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_{\Omega\setminus\Gamma} |\nabla g|^2 dx \quad + \quad \mathrm{Length}(\Gamma)$$

$\Downarrow$ $m(\Gamma) = 0$ $\qquad$ $\Updownarrow$ approximated

$$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$$

$$\Downarrow$$

$$\frac{\text{Convex M-S Energy}}{E(g)}$$

Mumford-Shah Model for SBV

Explanation of the approximation:

For functions of bounded variations ([Ambrosio-Giorgi, 88]), Mumford-Shah model becomes

$$\min_{g \in SBV} \left\{ \frac{\lambda}{2} \int_\Omega |f - g|^2 + \frac{\mu}{2} \int_{\Omega \setminus J_g} |\nabla g|^2 + \mathcal{H}^1(J_g) \right\},$$

where J_g is the jump set of g and $\mathcal{H}^1$ is the Hausdroff measure of dimension 1.

- See [Cagnetti & Scardia, 08] and [Strekalovskiy *et al.*, 12]
- If g is binary and piecewise-constant, then $J_g = \Gamma$ and

$$\mathcal{H}^1(J_g) = \boxed{\mathrm{Length}(\Gamma) = \int_\Omega |\nabla g|}$$

Convex Variant of the M-S Model

$$\frac{\text{Mumford-Shah Energy}}{E_{\mathrm{MS}}(g, \Gamma)}$$

Restrict:
$g \in W^{1,2}(\Omega)$

$\Updownarrow$

$$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx + \frac{\mu}{2}\int_{\Omega\setminus\Gamma} |\nabla g|^2 dx + \text{Length}(\Gamma)$$

$\downarrow$

$$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx + \frac{\mu}{2}\int_{\Omega} |\nabla g|^2 dx + \int_\Omega |\nabla g| dx$$

$\Updownarrow$

$$\frac{\text{Convex M-S Energy}}{E(g)}$$

Two-Stage Segmentation Method

$f \rightarrow$ stage 1 $\rightarrow$ Convex M-S Energy $E(g)$

$$\| $$

$$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx + \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx + \int_\Omega |\nabla g| dx$$

$\downarrow$

smooth solution $\hat{g}$

$\downarrow$

stage 2

$\downarrow$

Threshold $\hat{g}$ to piecewise constant

Stage One: Extension to Blur/Projected Problems

$$\frac{\text{Convex M-S Energy}}{E(g)}$$

stage 1

$$\Downarrow$$

$$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$$

$$\Big\downarrow \; f \leftarrow \mathcal{A}g + n$$

$$\frac{\lambda}{2}\int_\Omega (f-\mathcal{A}g)^2 dx \quad + \quad \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$$

Extendable to images corrupted by blur or projection $\mathcal{A}$.

Unique Minimizer for Stage One

Our *convex* variant of the Mumford-Shah model is:

$$\boxed{E(g) = \frac{\lambda}{2}\int_\Omega (f - \mathcal{A}g)^2 dx + \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx + \int_\Omega |\nabla g| dx}$$

Its discrete version is:

$$\boxed{\frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1}$$

Theorem

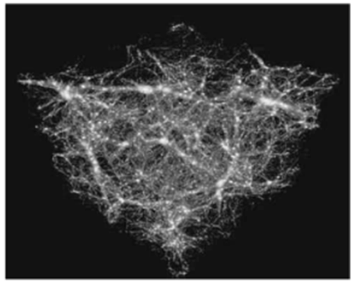
Let Ω be a bounded connected open subset of $\mathbb{R}^2$ with a Lipschitz boundary. Let $\boxed{\mathrm{Ker}(\mathcal{A}) \bigcap \mathrm{Ker}(\nabla) = \{0\}}$ *and $f \in L^2(\Omega)$, where $\mathcal{A}$ is a bounded linear operator from $L^2(\Omega)$ to itself. Then $E(g)$ has a unique minimizer $\hat{g} \in W^{1,2}(\Omega)$.*

Our Two-stage Segmentation Algorithm

Only 1 convex problem to solve

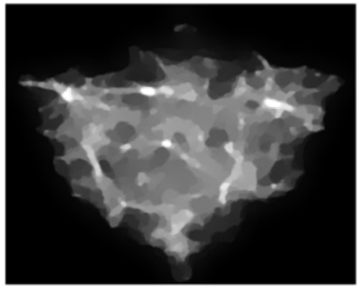
No iterations between Stages 1 and Stages 2

Given f



Stage 1: solve $\hat{g}$ in

$$\min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1 \right\}$$

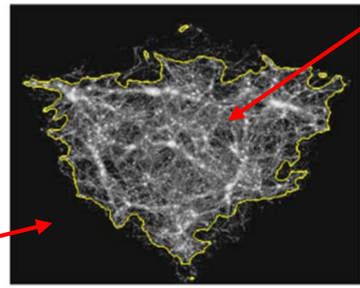


Stage 2: determine threshold ρ from $\hat{g}$ by mean or K-mean

ρ=0.19

K phases

$\hat{g} > 0.19$



$\hat{g} \leq 0.19$

Numerical Aspects: ***Stage One (Smoothing)***

Given f

Stage 1: solve $\hat{g}$ in

$$\min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1 \right\}$$

Stage 2: determine thresholds $\{\rho_i\}_{i=1}^{K-1}$ from $\hat{g}$

K phases

smoothing

solve by e.g. alternating direction method with multipliers (ADMM)

Numerical Aspects: Stage Two (Thresholding)

Given f

↓

Stage 1: solve $\hat{g}$ in

$$\min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1 \right\}$$

↓

Stage 2: determine thresholds $\{\rho_i\}_{i=1}^{K-1}$ from $\hat{g}$

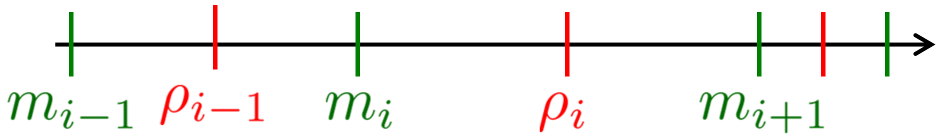
↓

K phases

thresholding

- ☐ 2-phase: mean of $\hat{g}$
- ☐ K-phase: use K-means on $\hat{g}$ to get K clusters and their centroids $\{m_i\}_{i=1}^{K}$; then set thresholds

$$\rho_i = \frac{m_{i+1} + m_i}{2}$$



Advantages of Smooth-&-threshold **(SaT)** *Method*

Given f

↓

Stage 1: solve $\hat{g}$ in

$$\min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1 \right\}$$

↓

Stage 2: determine thresholds $\{\rho_i\}_{i=1}^{K-1}$ from $\hat{g}$

↓

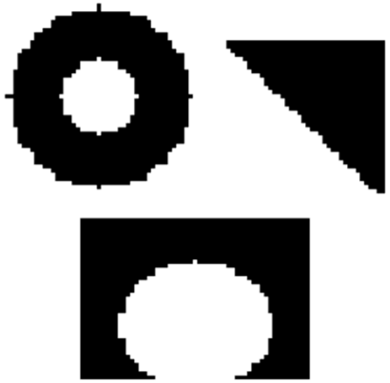
K phases

Advantages

- ☐ Stage 1 model for finding $\hat{g}$ is convex
- ☐ Stage 2 uses the same $\hat{g}$ when thresholds ρ_i or K change (No need to recompute $\hat{g}$)
- ☐ No need to fix K at the very beginning
- ☐ Easily adapted to different kinds of corruptions (e.g. blur, projection, non-Gaussian noise)

Motion Blurred and Noisy Image: Stage 1 Solution

2-phase image under vertical motion blur and Gaussian noise:

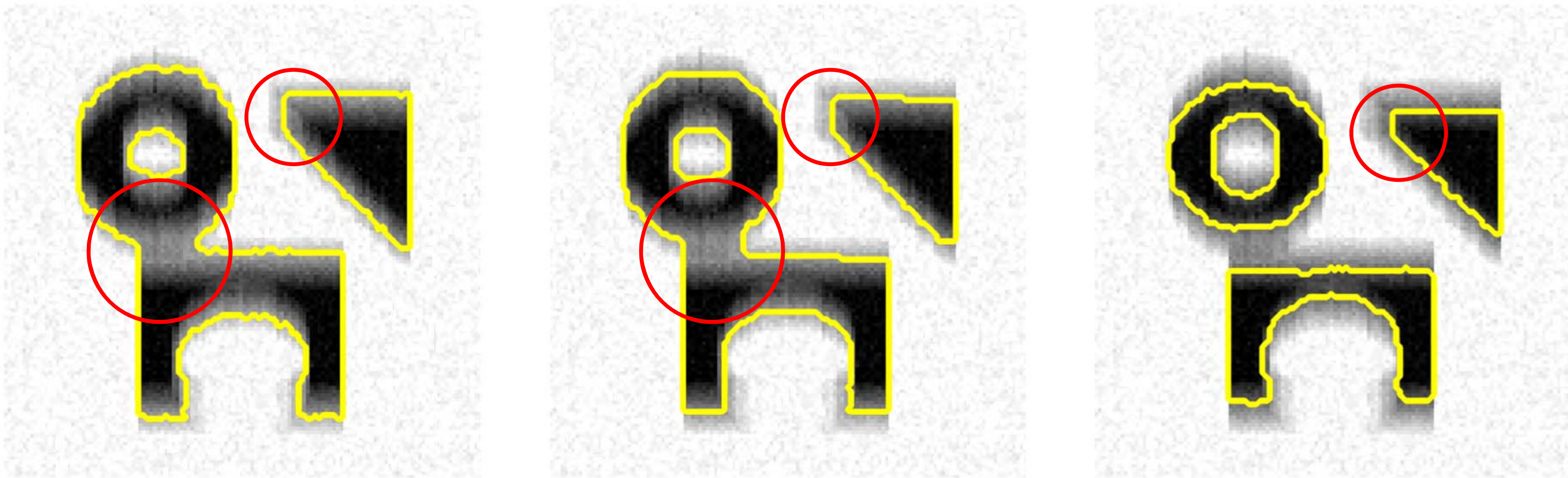


Given image

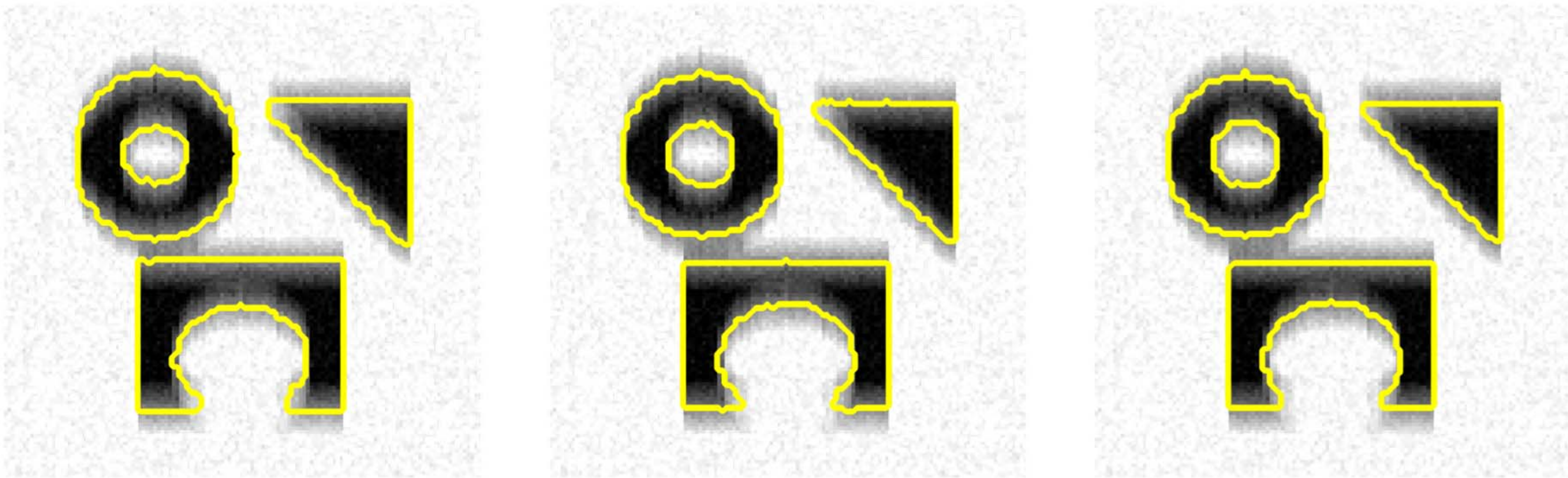
blurry & noisy image

Our solution $\hat{g}$

Motion Blurred and Noisy Image



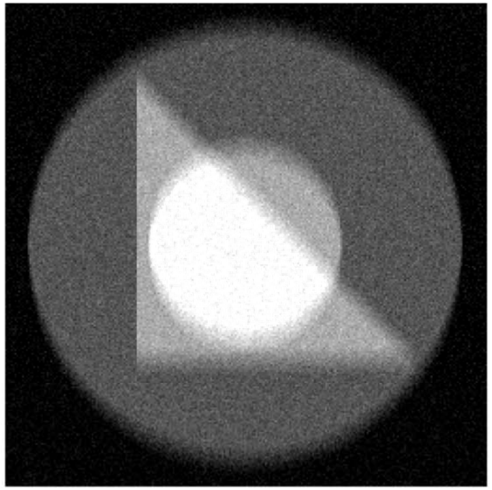
Chan-Vese (01) Dong *et al.* (10) Yuan *et al.* (10)



$\rho_{\text{mean}} = 0.7761$ $\rho_{\text{user}} = 0.6$ $\rho_{\{K=2\}} = 0.5048$

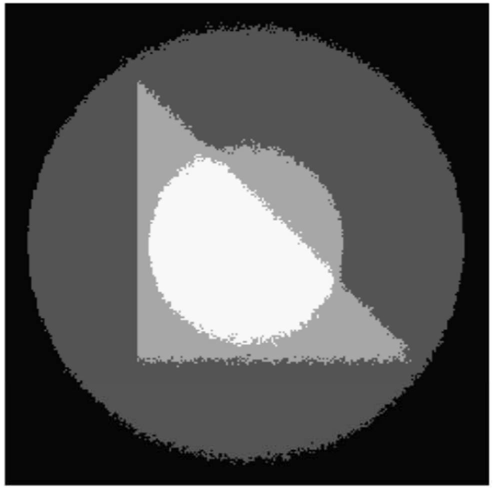
Robust with respect to the thresholds chosen

4-phase Segmentation of Noisy and Blurry Image

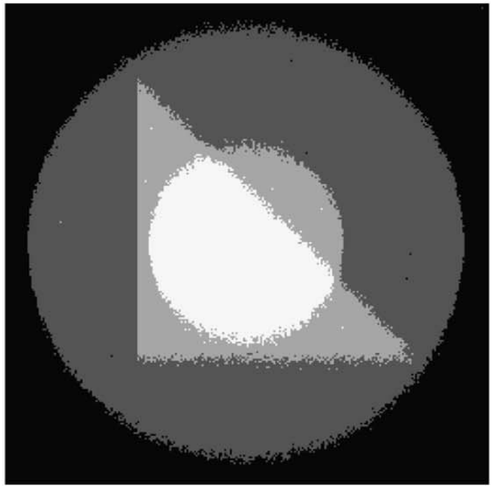


Noisy & blurry

Yuan *et al.* (10)

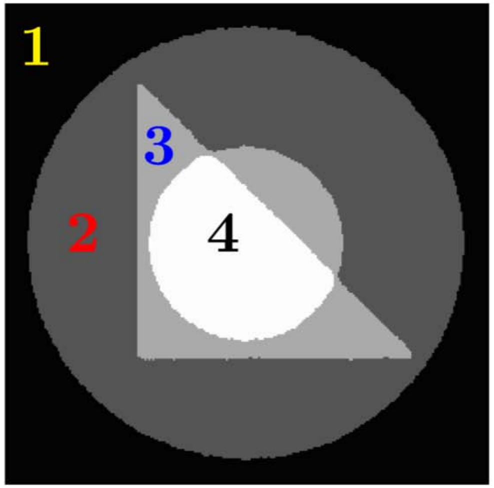


Li *et al.* (10)



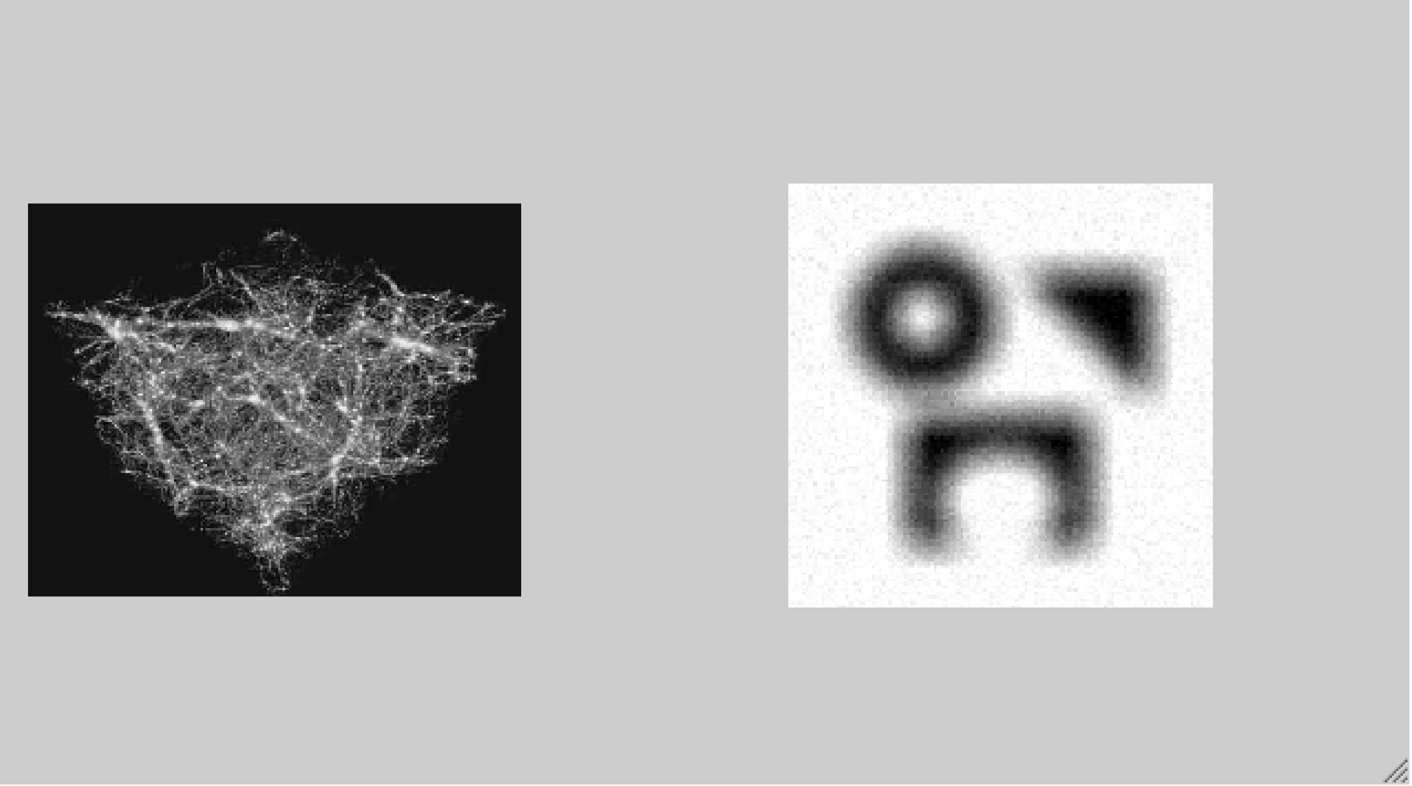
Sandberg *et al.* (10)

Steidl *et al.* (12)



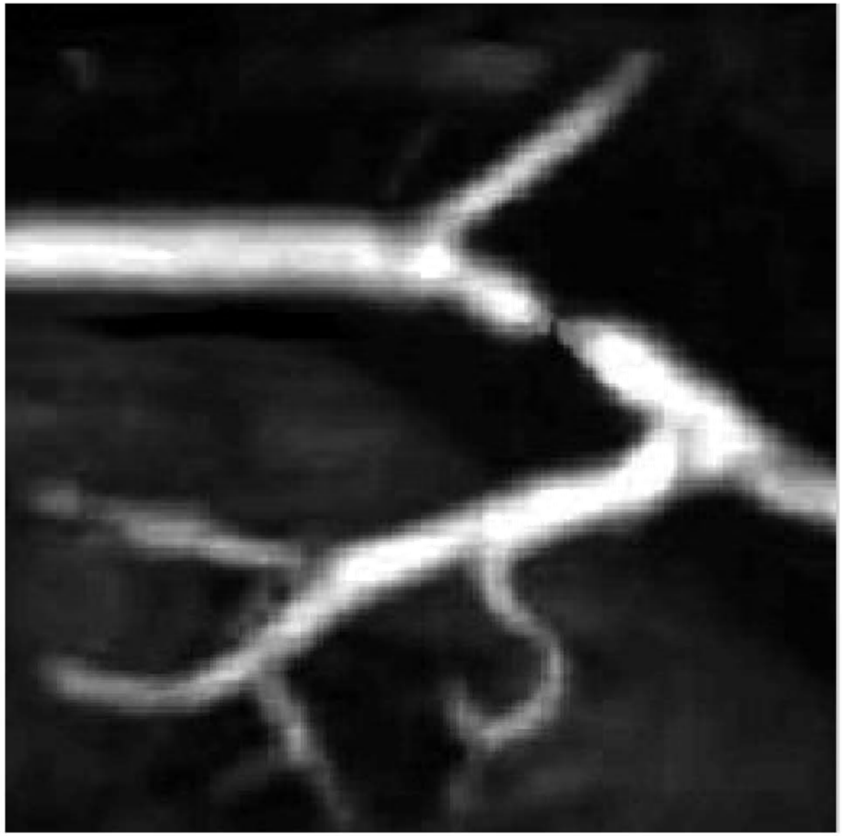
Our 4 phases from $\hat{g}$
using K-means ρ_i

Segmentation Changes with Threshold

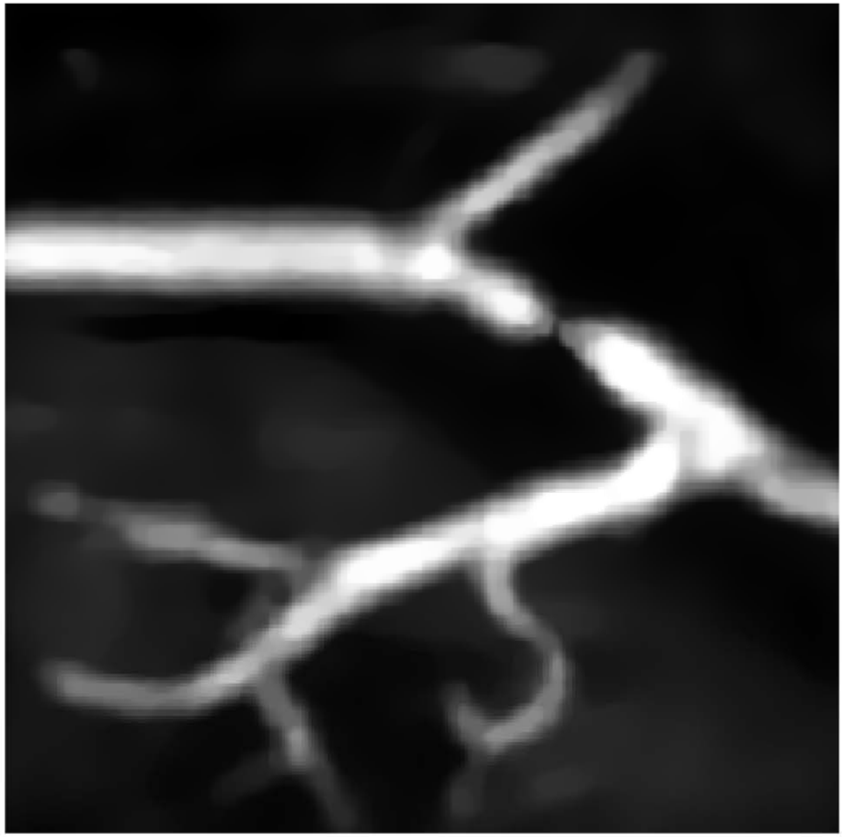


Γ changes as ρ changes. But no need to solve for $\hat{g}$ again. Just threshold $\hat{g}$ to get the phases.

Tubular MRA Image: Stage 1 Solution

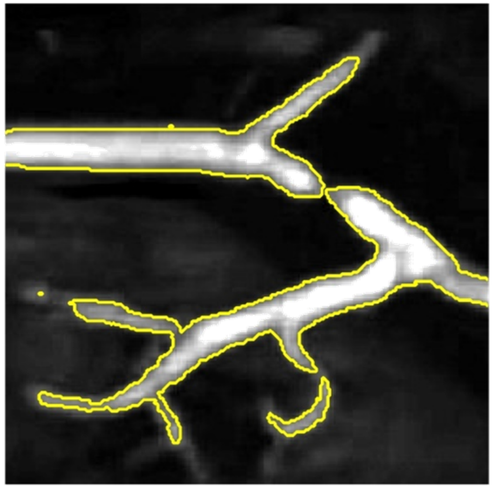


Given magnetic resonance
anigography image

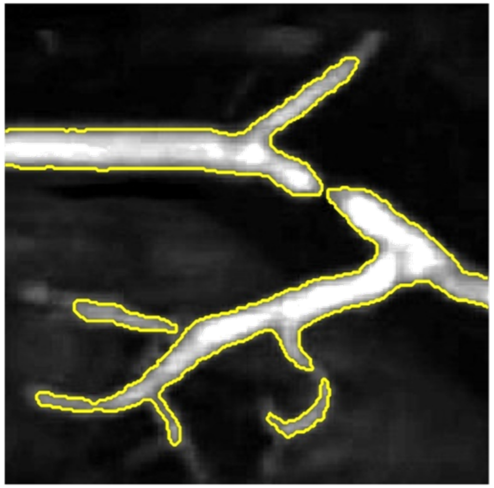


Our solution $\hat{g}$

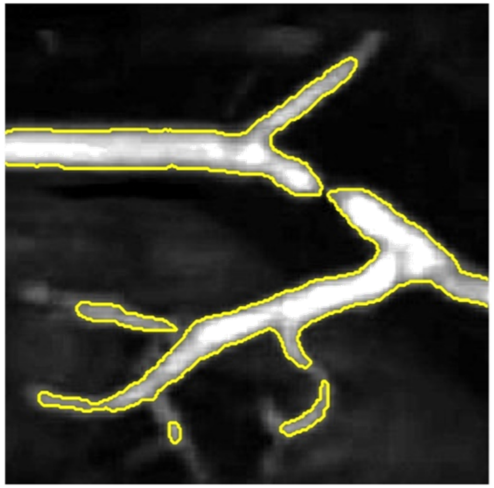
Tubular MRA Image



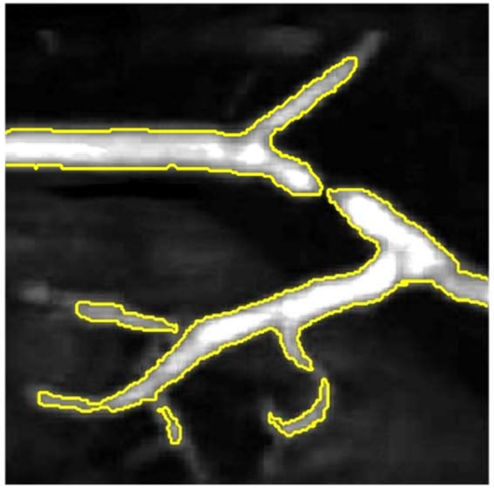
Chan-Vese (01)



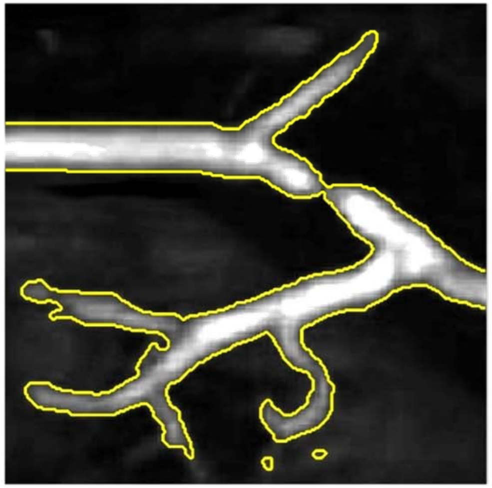
Yuan *et al.* (10)



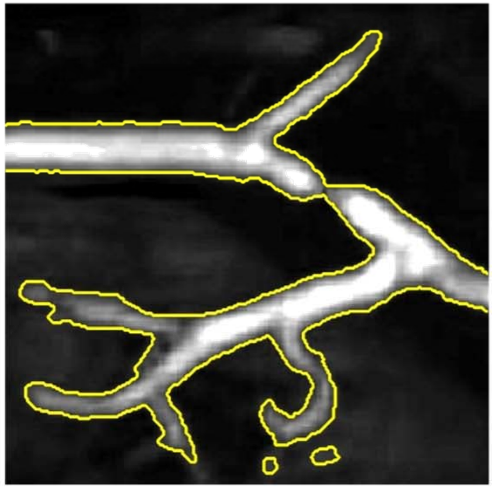
$\rho_{\{K=2\}} = 0.4019$



Dong *et al.* (10)



Cai *et al.* (13)



$\rho_{\text{mean}} = 0.1760$

Cai, C., and Zeng, SIAM J. Imag. Sci. (2013)

Segmentation under Poisson or Gamma Noise

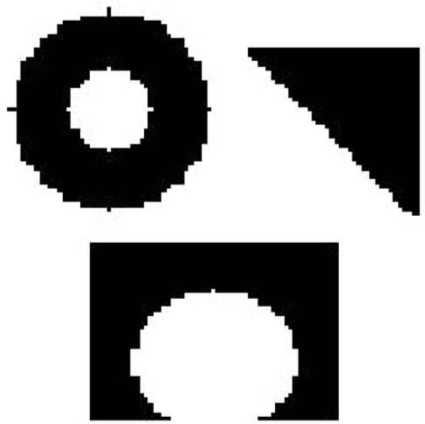
First stage: given f, solve

$$\min_g \left\{ \lambda \int_\Omega (\mathcal{A}g - f \log \mathcal{A}g)dx + \frac{\mu}{2} \int_\Omega |\nabla g|^2 dx + \int_\Omega |\nabla g| dx \right\}.$$

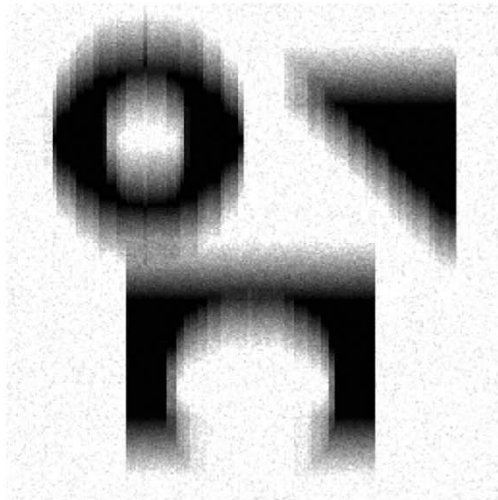
- data fitting term good for Poisson noise from MAP analysis
- also good for multiplicative Gamma noise (Steidl and Teuber (10))
- objective functional is convex (solved by Chambolle-Pock)
- admits unique solution $\hat{g}$ if $\mathrm{Ker}(\mathcal{A}) \cap \mathrm{Ker}(\nabla) = \{0\}$

Second stage: threshold the solution $\hat{g}$ to get the phases.

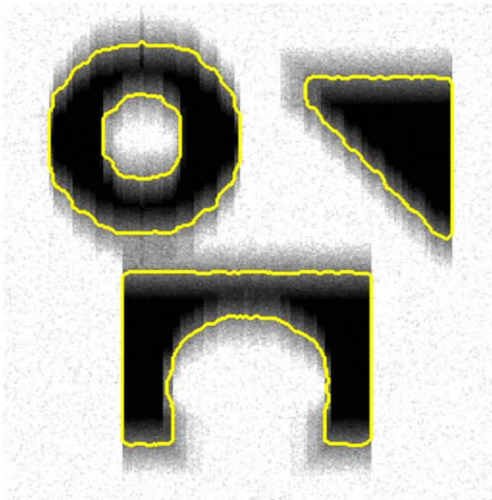
3-object Image with Poisson Noise and Motion Blur



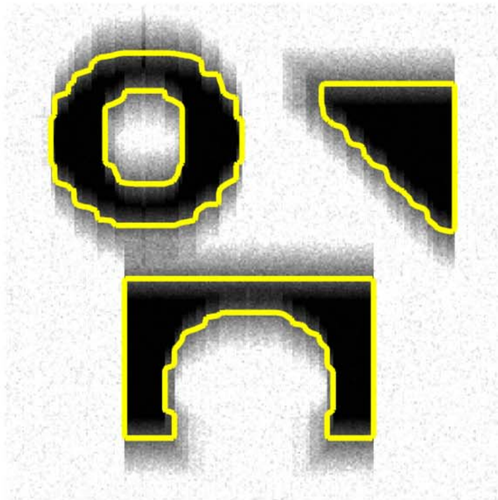
Original image



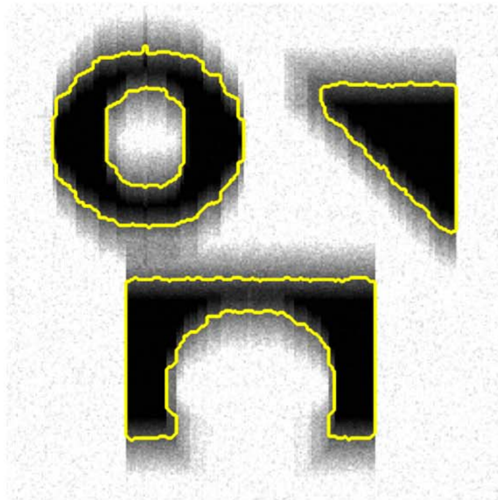
Noisy & blurred



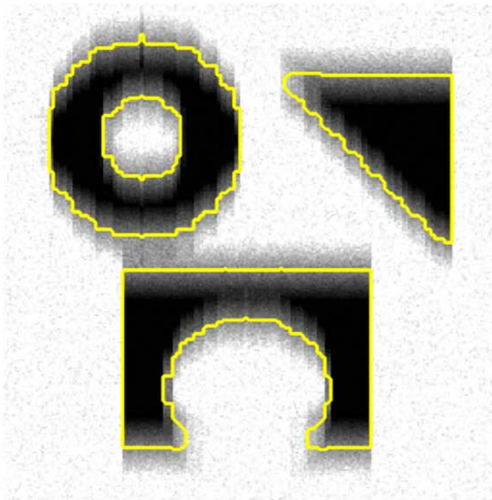
Yuan *et al.* (10)



Dong *et al.* (10)

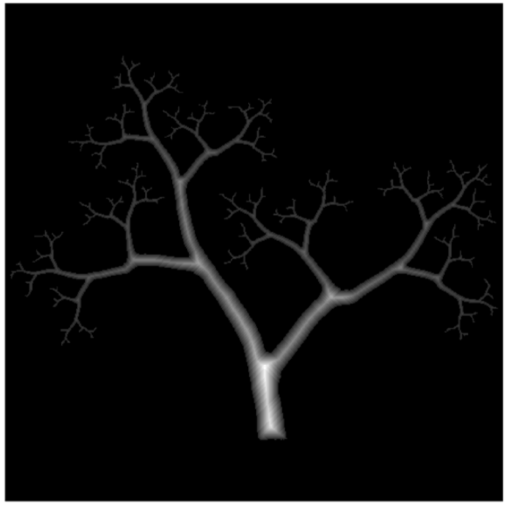


Sawatzky *et al.* (13)

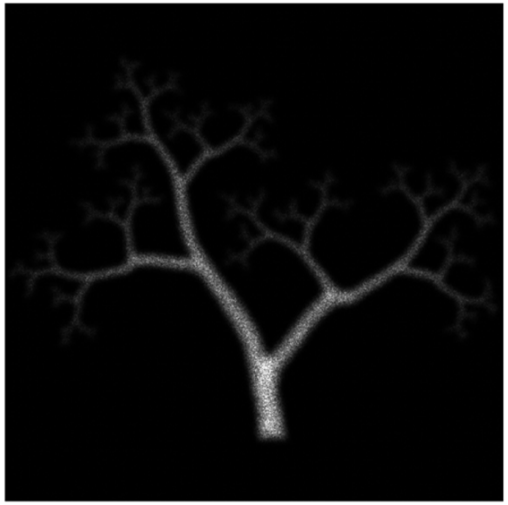


$\rho_{\{K=2\}} = 129.94$

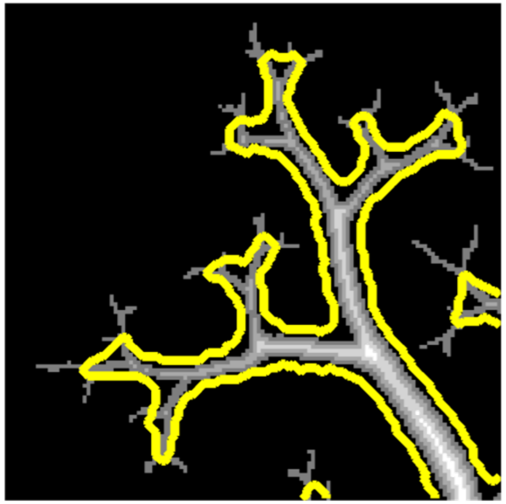
Fractal Tree with Gamma Noise and Gaussian Blur



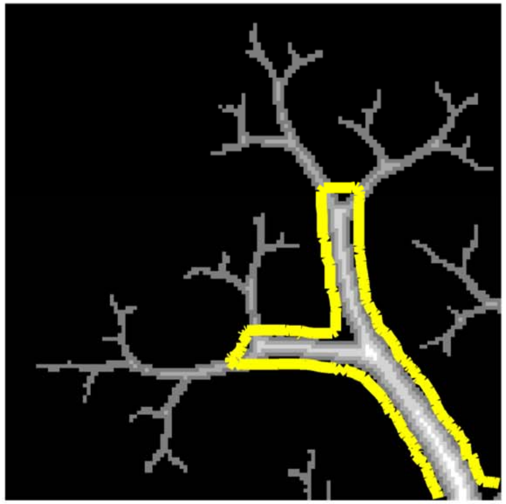
Original image



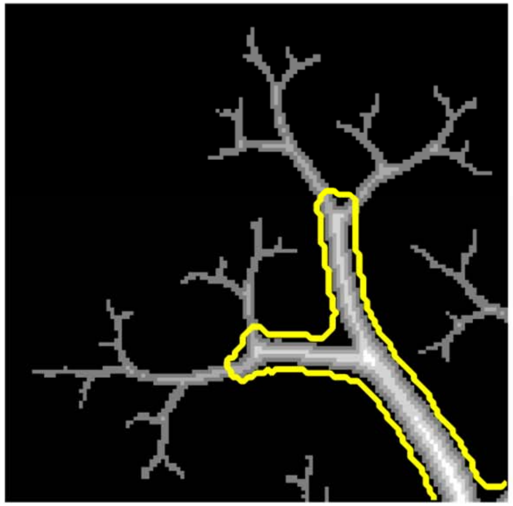
Noisy & blurred



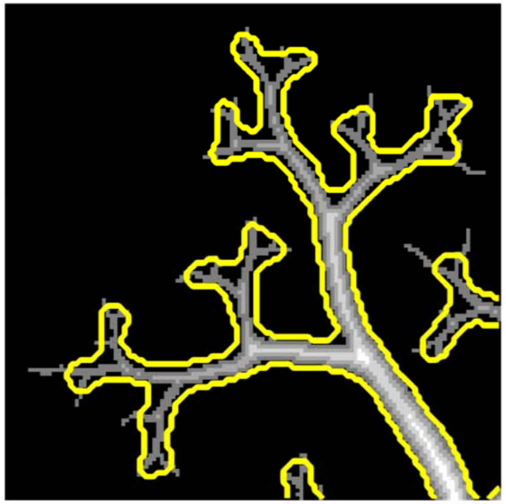
Yuan *et al.* (10)



Dong *et al.* (10)



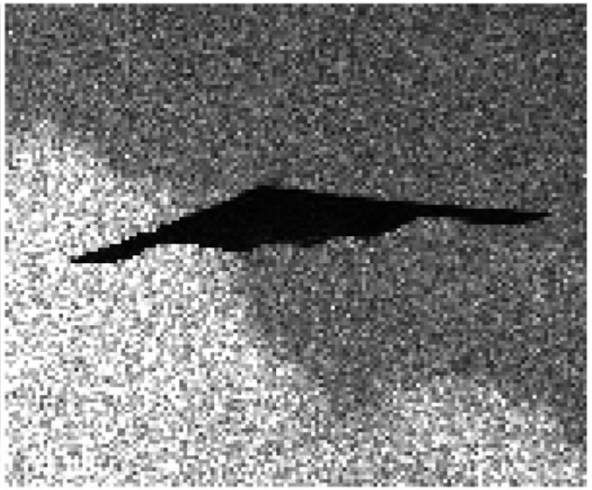
Sawatzky *et al.* (13)



$\rho_{\text{user}} = 14$

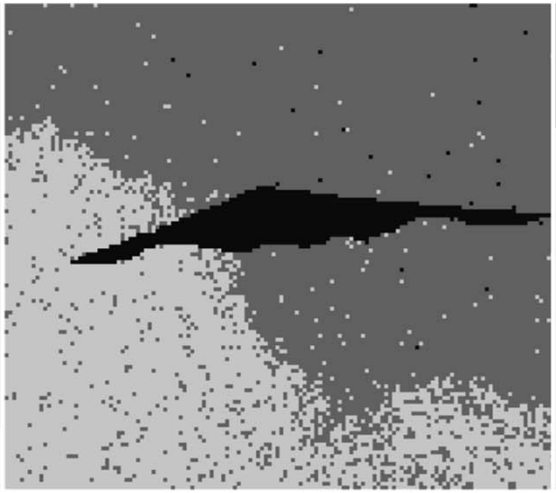
Airplane with Multiplicative Gamma Noise

Original image

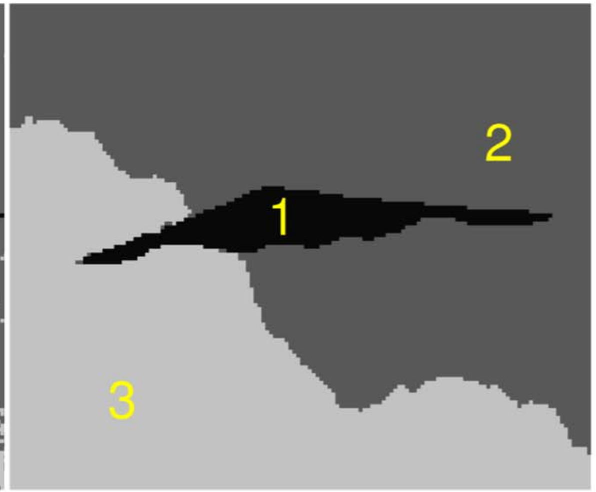


Noisy image

Yuan *et al.* (10)

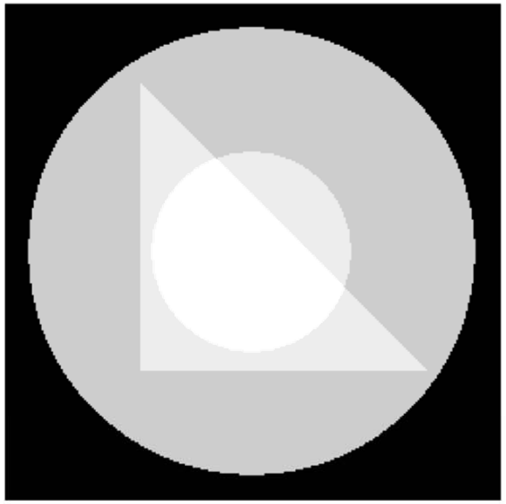


Li *et al.* (10)

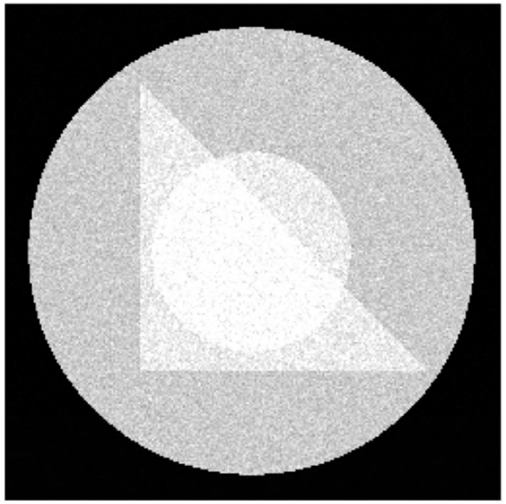


SaT with $\{\rho_i\}_{i=1}^{2}$ from K-means

4-phase with Close Intensity under Poisson Noise



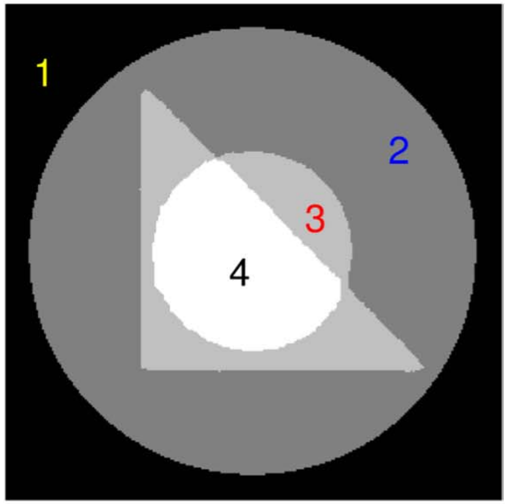
Original image



Poisson noise

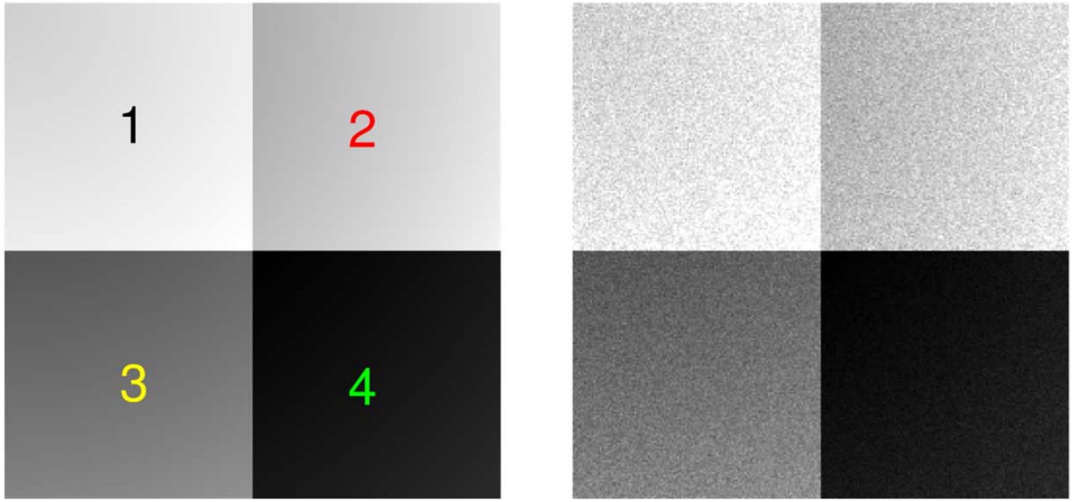
Yuan *et al.* (10)

Li *et al.* (10)



SaT with $\{\rho_i\}_{i=1}^{3}$ from K-means

Image with Close and Varying Intensities



Original image   Poisson noise

Yuan *et al.* (10)   Li *et al.* (10)   SaT with $\{\rho_i\}_{i=1}^{3}$ from K-means

CPU Time

Multi-phase: iteration numbers and CPU time in second

Test	Yuan		Li		Our Method	
	iter.	time	iter.	time	iter.	time
Airplane	127	1.0	95	1.0	86	0.2
4-phase	57	2.2	49	1.6	184	2.3
Close-intensity	34	1.8	110	4.0	84	0.5
Varying-intensity	114	4.4	332	9.9	444	3.0
MRA	76	25.7	114	26.4	19	0.6

C., Yang, and Zeng, SIAM J. Imag. Sci. (2014)

Outline

1. **Mumford-Shah Model**
2. **Two-stage Image Segmentation Method**
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Is 2-stage Enough for Color Images?

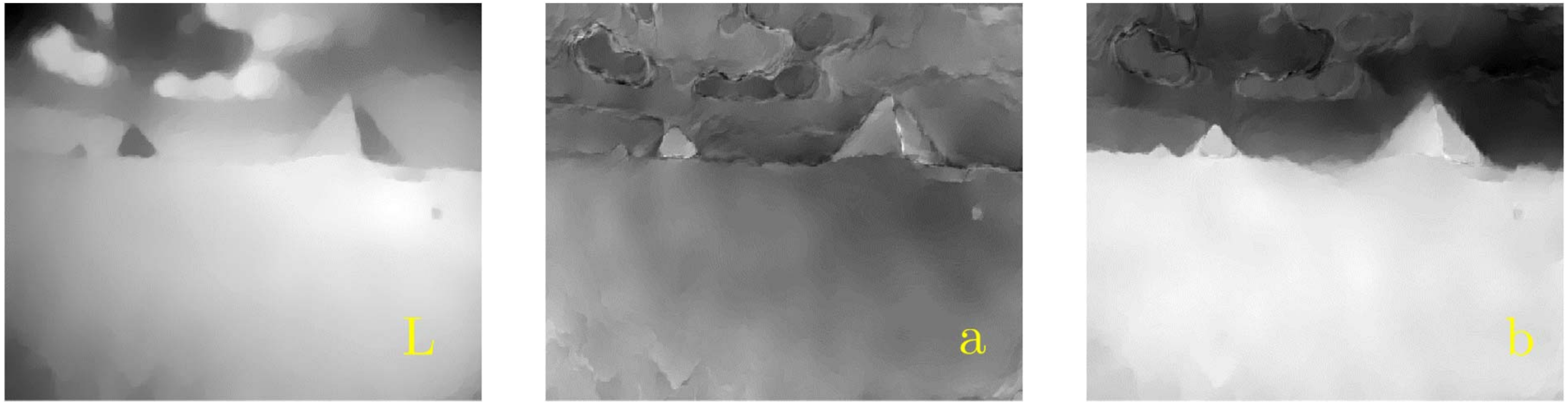
noisy image

K-mean thresholding

RGB: strong inter-channel correlation

Less-correlated Color Space

RGB: strong inter-channel correlation



Lab channels: less correlated

Thresholding using all six-channels

Three-stage (SLaT) *Method for Color Images*

Stage 1 (smoothing): given $f = (f_1, f_2, f_3)$, solve

$$\min_{g_i} \left\{ \lambda \int_\Omega (\mathcal{A} g_i - f_i)^2 dx + \frac{\mu}{2} \int_\Omega |\nabla g_i|^2 dx + \int_\Omega |\nabla g_i| dx \right\}, \; i = 1, 2, 3,$$

to obtain smooth unique solution $\hat{g} = (\hat{g}_1, \hat{g}_2, \hat{g}_3)$.

Stage 2 (lifting):

- ☐ transform $\hat{g}$ to another color space $\bar{g} = (\bar{g}_1, \bar{g}_2, \bar{g}_3)$ with less-correlation among the channels
- ☐ Then form the uplifted image $g = (\hat{g}_1, \hat{g}_2, \hat{g}_3, \bar{g}_1, \bar{g}_2, \bar{g}_3)$

Stage 3 (thresholding): Use K-means to threshold uplifted image g to get the phases.

2-phase Segmentation for Noisy Color Image

Clean image

Noisy image

Li *et al.* (10)

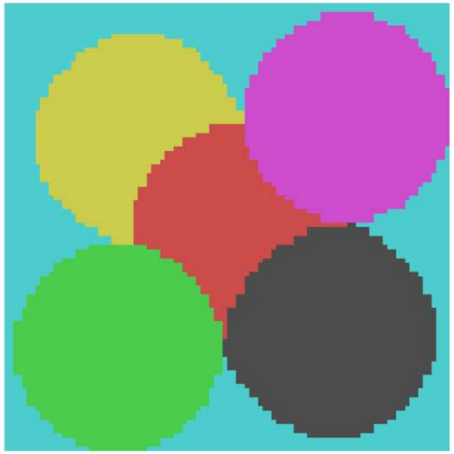
Pock *et al.* (09)

Strorath *et al.* (14)

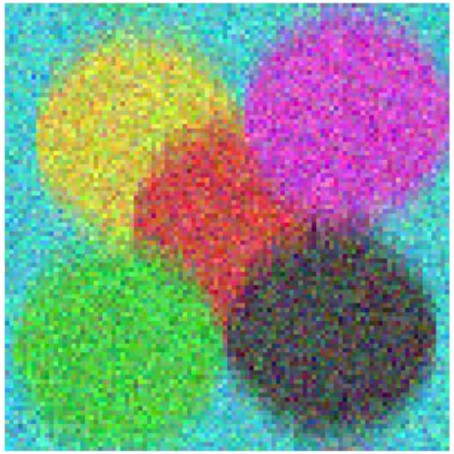
SaT with thresholds from K-means

Gaussian noise with s.d. 0.1.

6-phase Segmentation for Noisy & Blurry Image

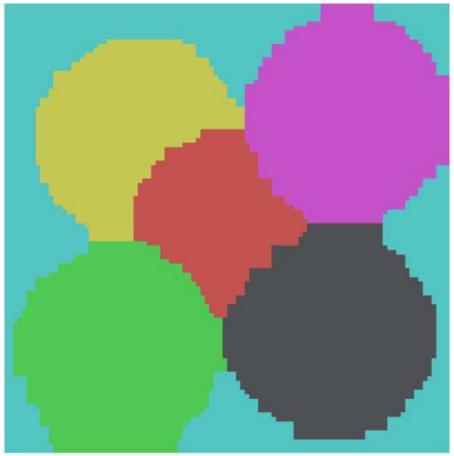


Clean image



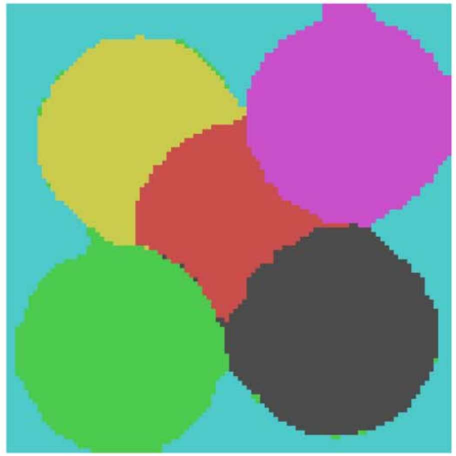
blurry & noisy

Li *et al.* (10)



Pock *et al.* (09)

Strorath *et al.* (14)



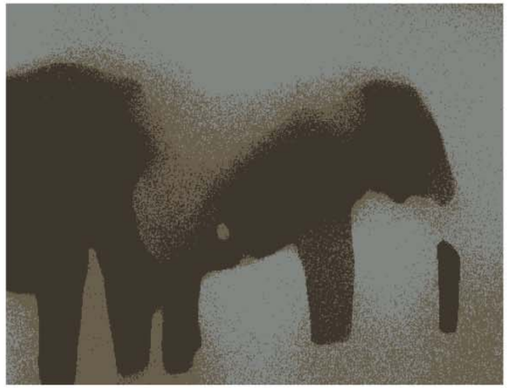
SaT with thresholds from K-means

10-pixel vertical motion blur with Gaussian noise.

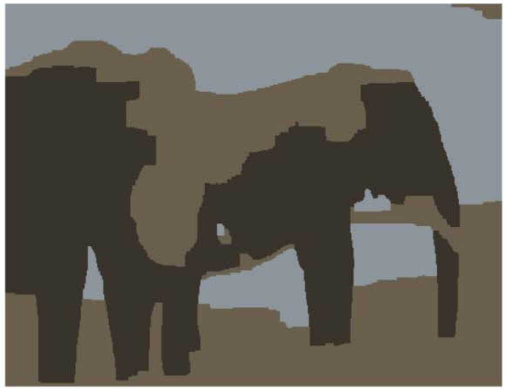
3-phase Segmentation for Noisy & Blurry Image

Clean image

blurry & noisy



Li *et al.* (10)



Pock *et al.* (09)

Strorath *et al.* (14)

SaT with thresholds from K-means

10-pixel vertical motion blur with Poisson noise added

4-phase Segmentation for Pixel-loss Color Image

Clean image

Noisy image

Li *et al.* (10)

Pock *et al.* (09)

Strorath *et al.* (14)

SaT with $\rho_{K=4}$

60% pixel loss with Poisson noise added

Cai, C., Nikolova, and Zeng, J. Sci. Comput., (2017)

Outline

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Smooth-and-Threshold (SaT) *Method*

$$\frac{\text{Convex M-S Energy}}{E(g)} \longleftarrow \text{stage 1} \longleftarrow f$$

$$\|$$

$$\frac{\lambda}{2}\int_\Omega (f - \mathcal{A}g)^2 dx \quad + \quad \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$$

$$\downarrow$$

smooth solution $\hat{g}$

$$\downarrow$$

stage 2

$$\downarrow$$

Threshold $\hat{g}$ to
piecewise constant

Relationship with Image Restoration

Convex M-S Energy

$$E(g)$$

$$\Vert$$

$$\frac{\lambda}{2}\int_{\Omega}(f - \mathcal{A}g)^2 dx \;+\; \cancelto{0}{\frac{\mu}{2}\int_{\Omega}|\nabla g|^2 dx} \;+\; \int_{\Omega}|\nabla g|\, dx$$

ROF Model (1992)
Rudin, Osher and Fatemi

staircase

reduce by introducing higher-order derivative:

T. Chan (00), Lysaker (03), Steidl (08), Bredies (10), Hintermüller (06), etc.

Relationship with Image Restoration

$$\frac{\lambda}{2}\int_\Omega (f - \mathcal{A}g)^2 dx \quad + \quad \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$$

$$\Downarrow$$

Hintermüller (2006): image restoration model

restoration + thresholding = segmentation

Cai & Steidl, EMMCVPR, 2013:

ROF model + thresholding = Chan-Vese

Non-convex Regularizers

$$\frac{\text{Convex M-S Energy}}{E(g)}$$

$$\Updownarrow$$

$$\frac{\lambda}{2}\int_\Omega (f - \mathcal{A}g)^2 dx + \underbrace{\frac{\mu}{2}\int_\Omega |\nabla g|^2 dx + \int_\Omega |\nabla g| dx}_{\text{non-convex regularizer}}$$

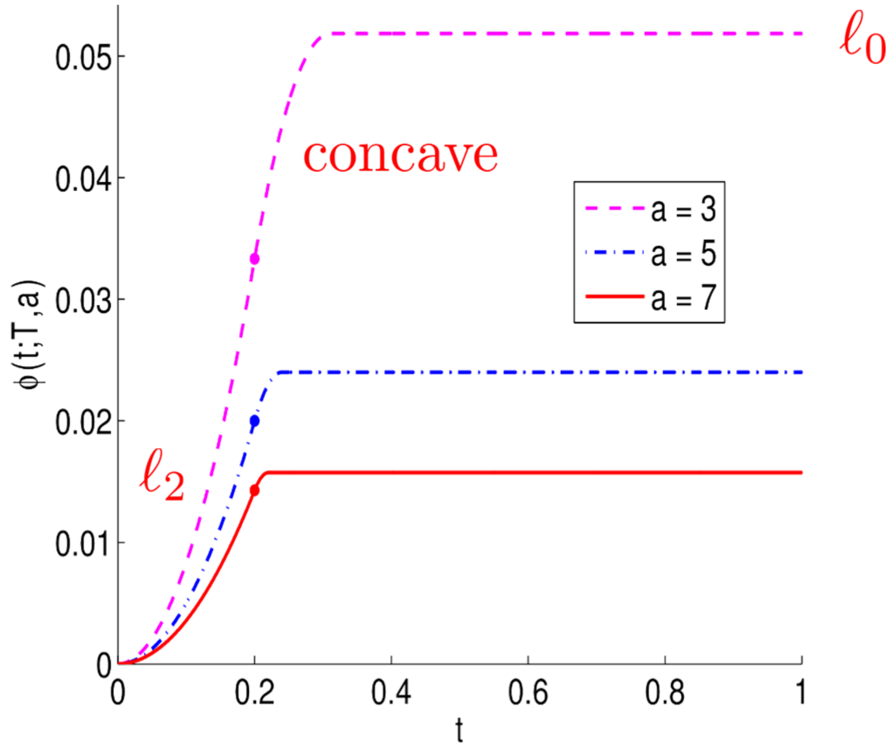
non-convex regularizer → better restoration model

non-convex regularizer → better segmentation method → computational cost?

Our Non-convex Regularizer

$$\phi(t;T,a) := \begin{cases} \frac{a(T_2-T)}{2T}t^2 & t \in [0,T) \\ -\frac{a}{2}t^2 + aT_2 t - \frac{aTT_2}{2} & t \in [T,T_2) \\ \frac{aT_2(T_2-T)}{2} & t \in [T_2,\infty) \end{cases}$$

with $T_2 = T + \frac{1}{a^2}$



Convex Non-convex Model

Convex Non-convex Energy

$\mathrm{CNC}(g)$

$$\underbrace{\frac{\lambda}{2}\int_{\Omega}(f-g)^2dx + \underbrace{\sum_{i=1}^{n}\phi\left(\|(\nabla g)_i\|_2; T, a\right)}_{\text{non-convex regularizer}}}_{\text{convex functional}} \quad \text{if } \lambda > 9a$$

C., Lanza, Morigi, Sgallari, Num. Math., 2018

Two-Stage Convex Non-convex Segmentation

f → stage 1 → $\dfrac{\text{Convex Non-convex Energy}}{\text{CNC}(g)}$

$$\text{CNC}(g) = \frac{\lambda}{2}\int_\Omega (f-g)^2 dx + \sum_{i=1}^{n} \phi\left(\|(\nabla g)_i\|_2; T, a\right)$$

↓

smooth solution $\hat{g}$ by ADMM

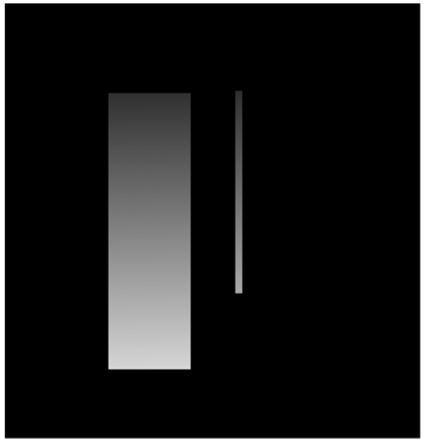
↓

stage 2

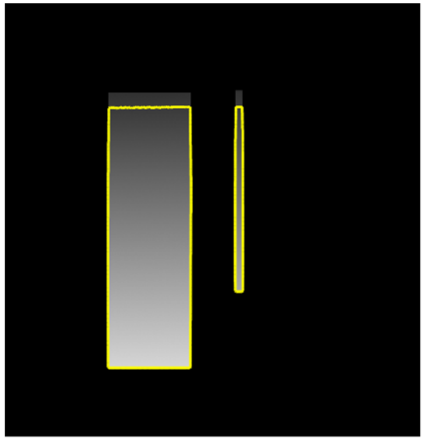
↓

Threshold $\hat{g}$ to
piecewise constant

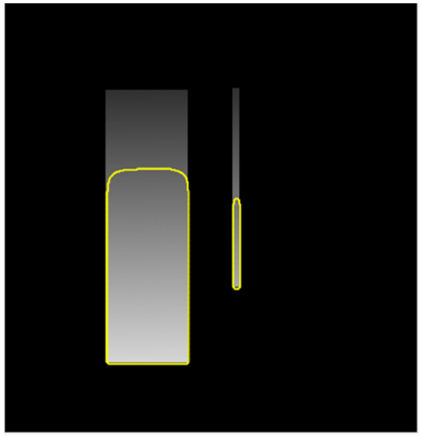
Convex Non-convex SaT Segmentation Method



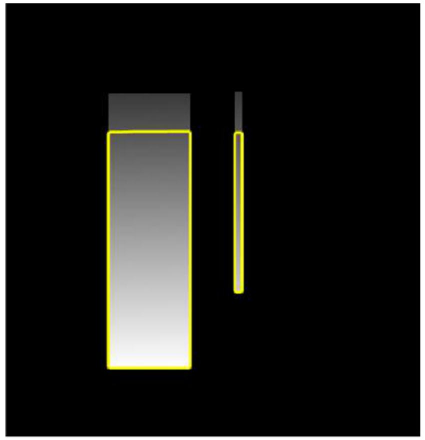
given image



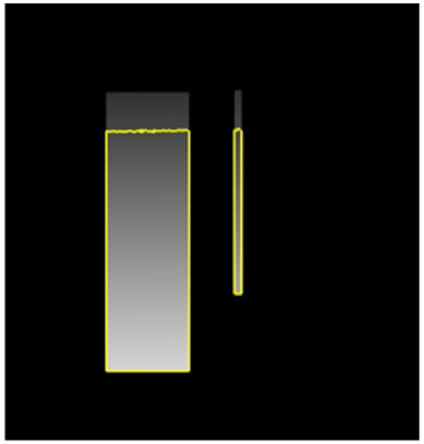
Chan-Vese



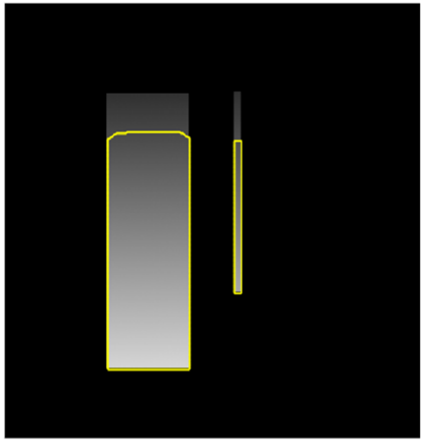
Bae *et al.*



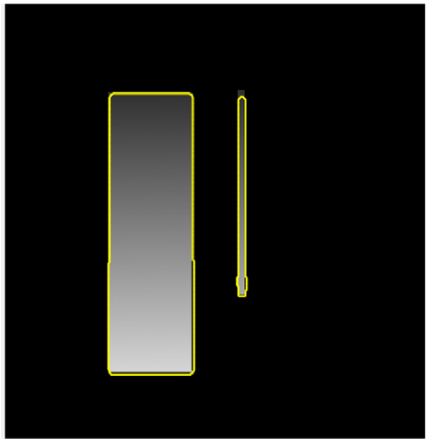
Dong *et al.*



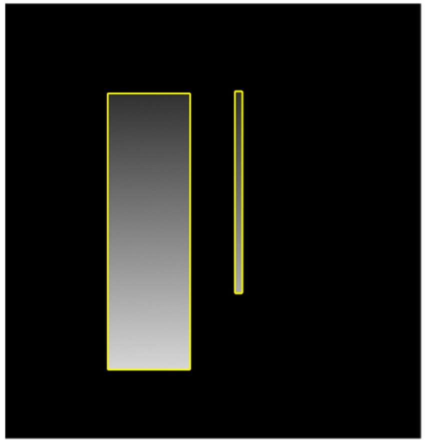
Sandberg *et al.*



convex SaT
+ K-means

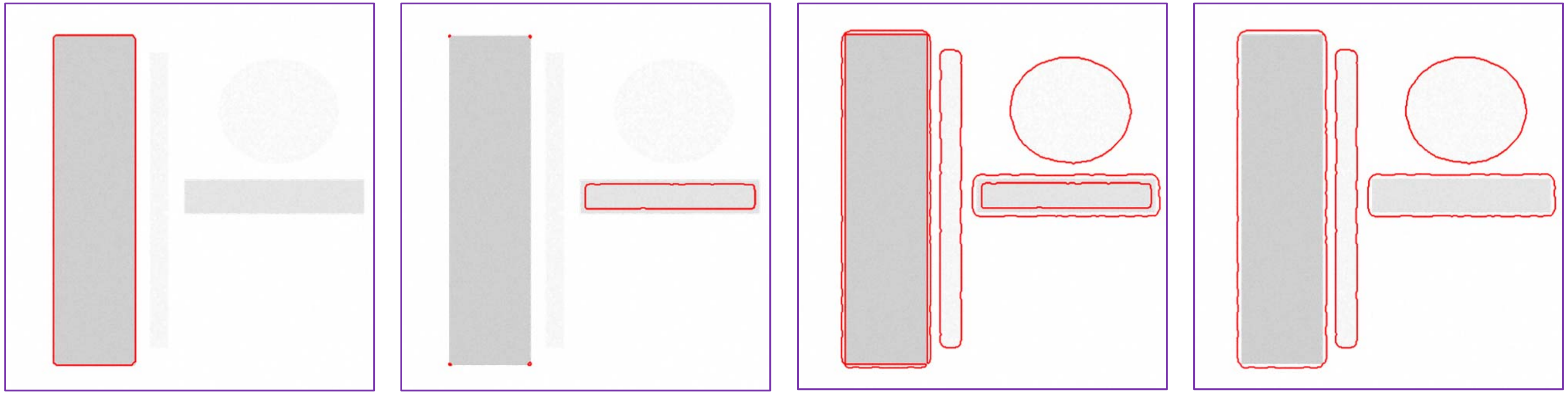


convex SaT
with $\rho = .19$

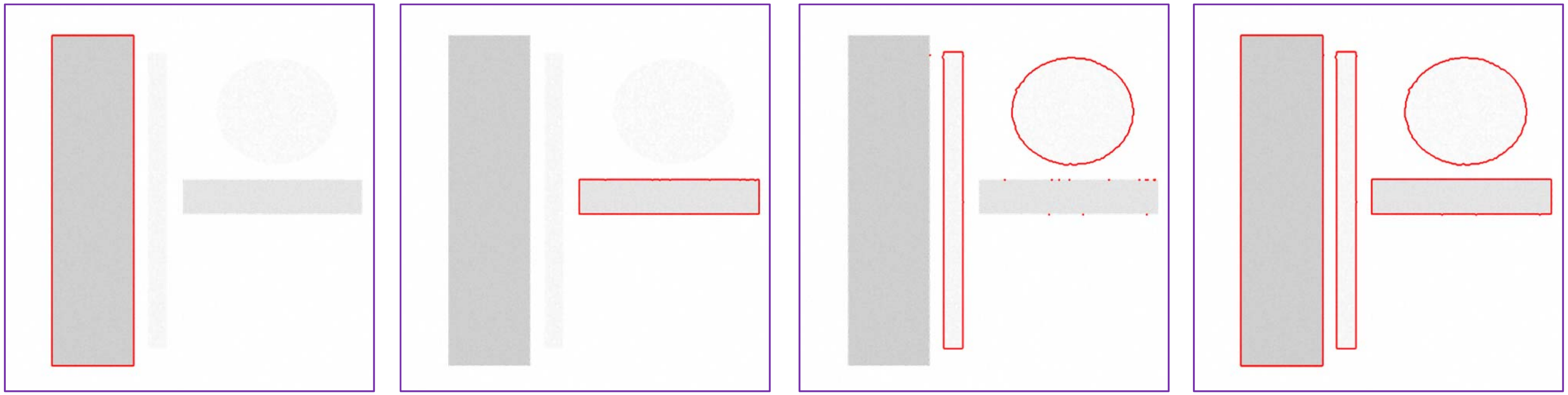


convex non-
convex SaT

4-phase with Close Intensity



Regions 1 to 4 by convex SaT



Regions 1 to 4 by convex non-convex SaT

Outline

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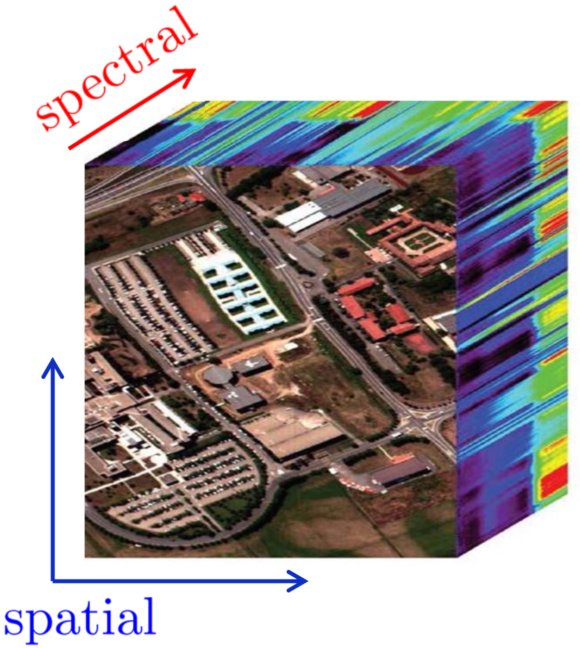
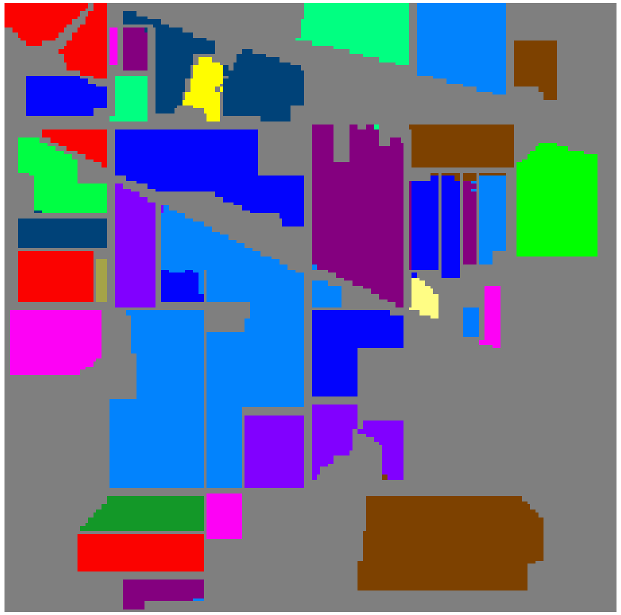
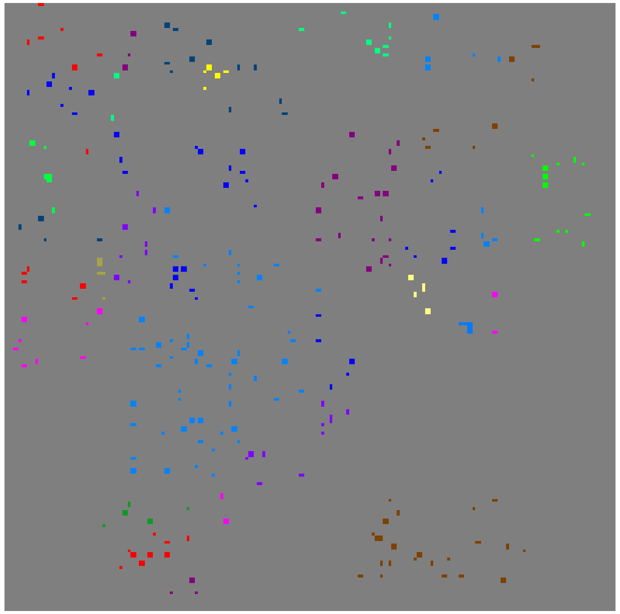
Hyper-spectral Image Classification

analyze the material for each pixel





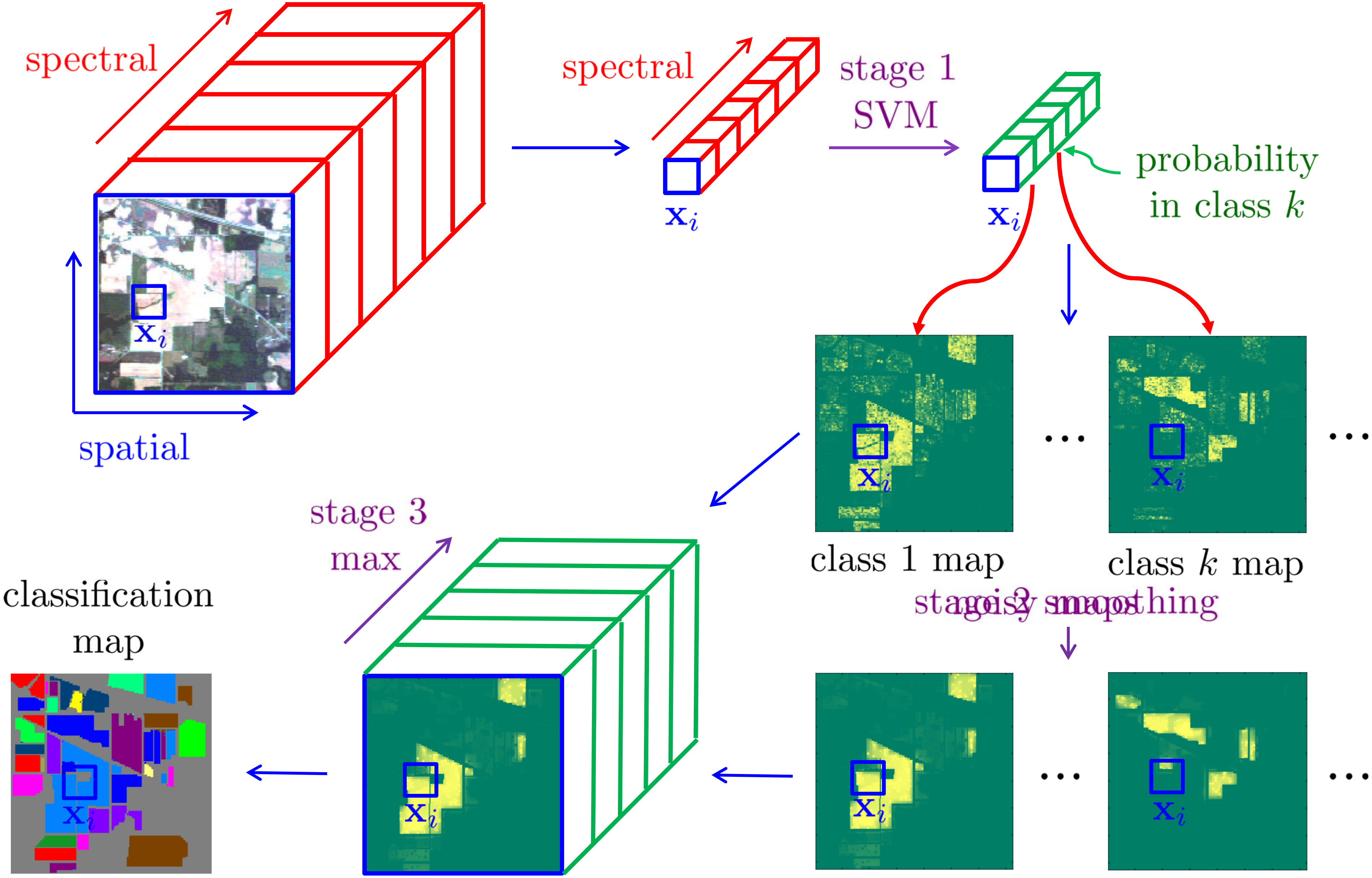
Our method



Training pixels
10% = 1048 pixels

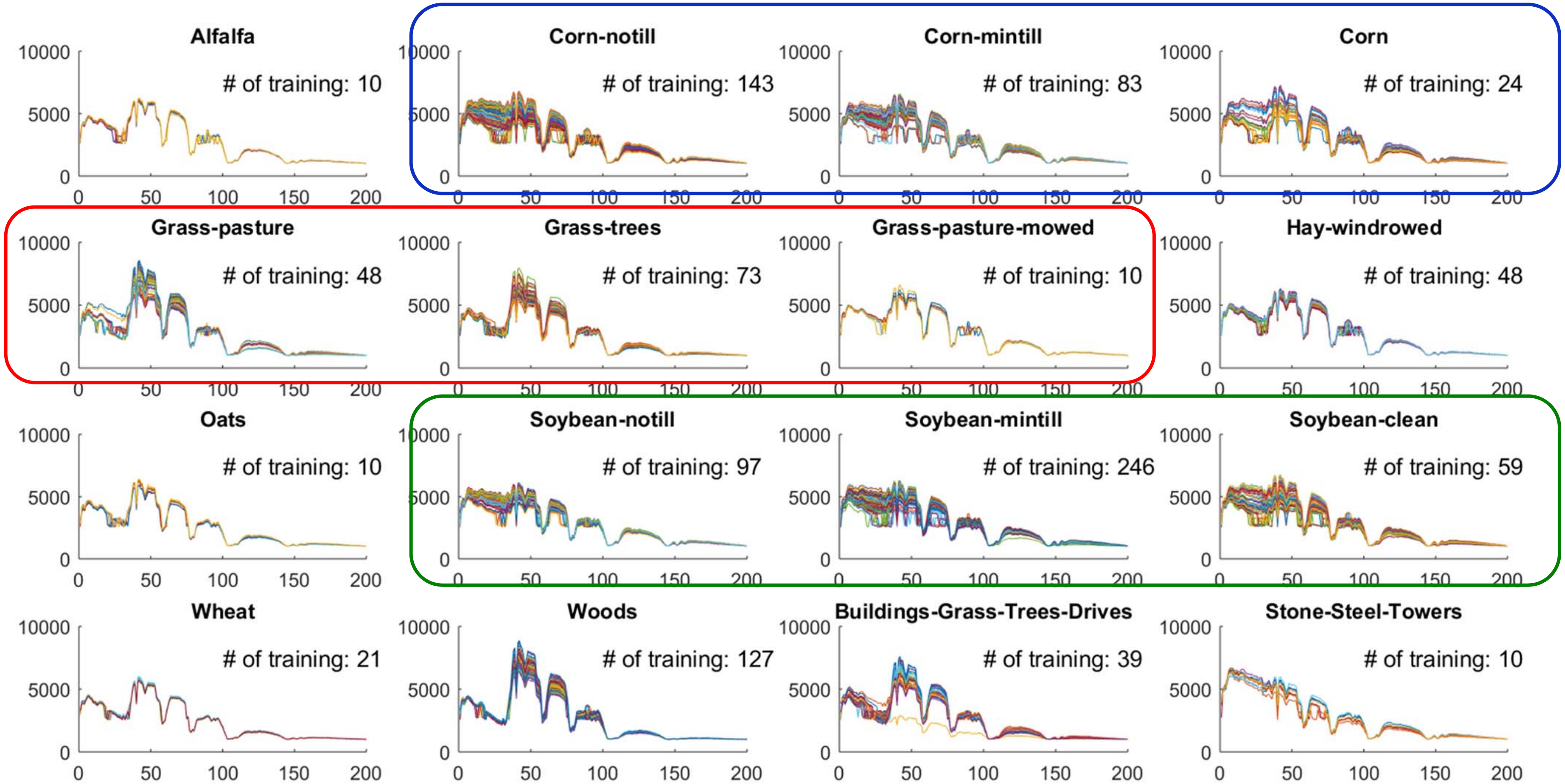
Our Smooth-and-Threshold (SaT) *Approach*

Traditional SVM-Method



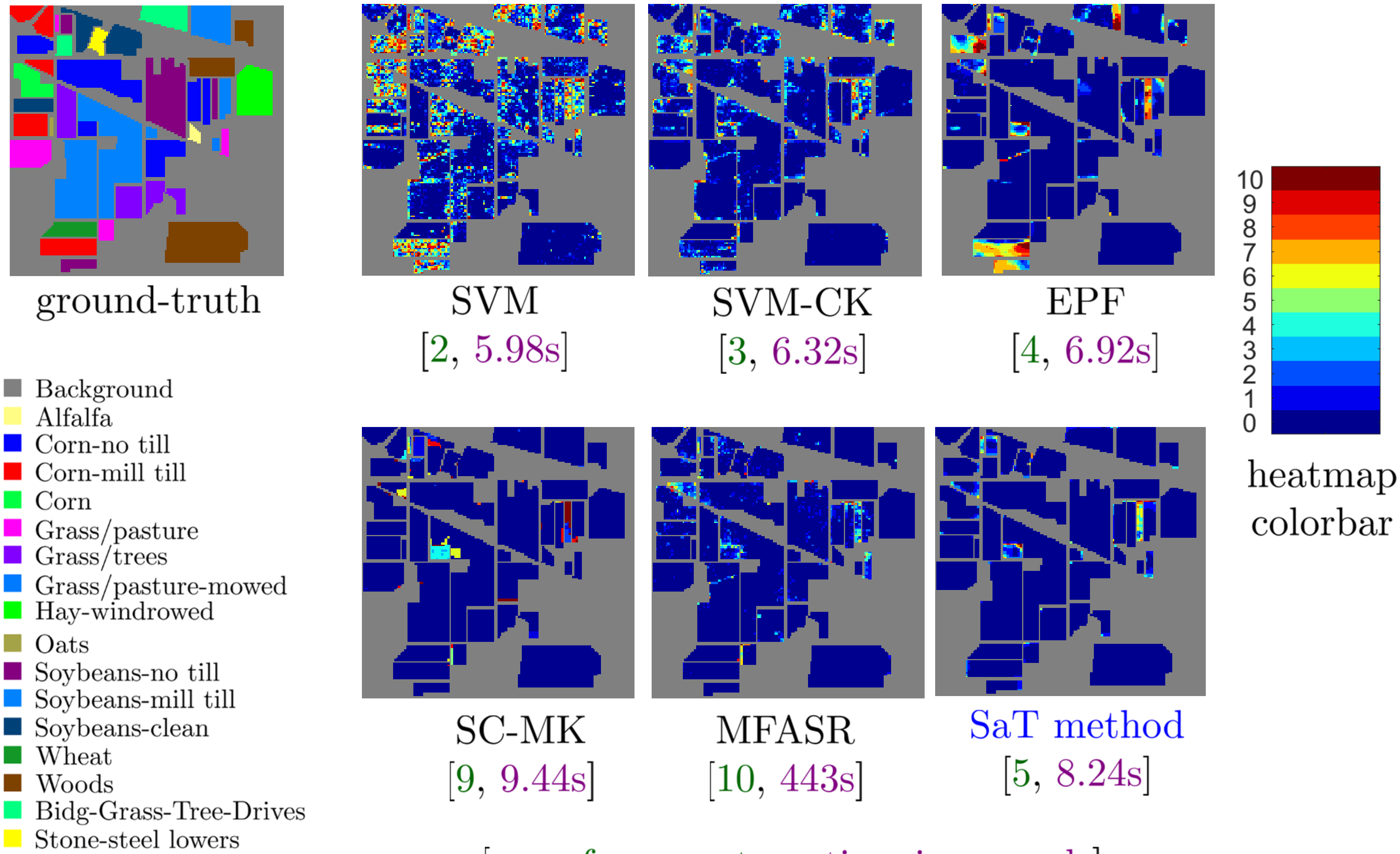
Indian Pines Data Set

- Data size: 145×145 (spatial) $\times$ 200 (spectral)
- Close spectrum between classes



Indian Pines Data Set

Error heat map over 10 trials with random 10% training pixels



label color

[no. of parameters, time in seconds]

Comparison with Other Methods

Accuracy over 10 random trials

	SVM	SVM-CK	EPF	SC-MK	MFASR	SaT	gain
overall accuracy	79.78%	92.11%	93.34%	97.83%	97.88%	98.83%	0.95%
average accuracy	80.11%	92.68%	95.95%	98.35%	97.91%	98.88%	0.35%
kappa	76.90%	91.01%	92.36%	97.52%	97.58%	98.66%	1.08%

- overall accuracy: percentage of correctly classified pixels
- average accuracy: average of the accuracy in each class
- kappa: Cohen's kappa coefficient

SVM [Melgani *et al.*, 2004], SVM-CK [Camps-Valls *et al.*, 2006],
EPF [Kang *et al.*, 2014], SC-MK [Fang *et al.*, 2015],
MFASR [Fang *et al.*, 2017].

Effect of the High-order Smoothing Term

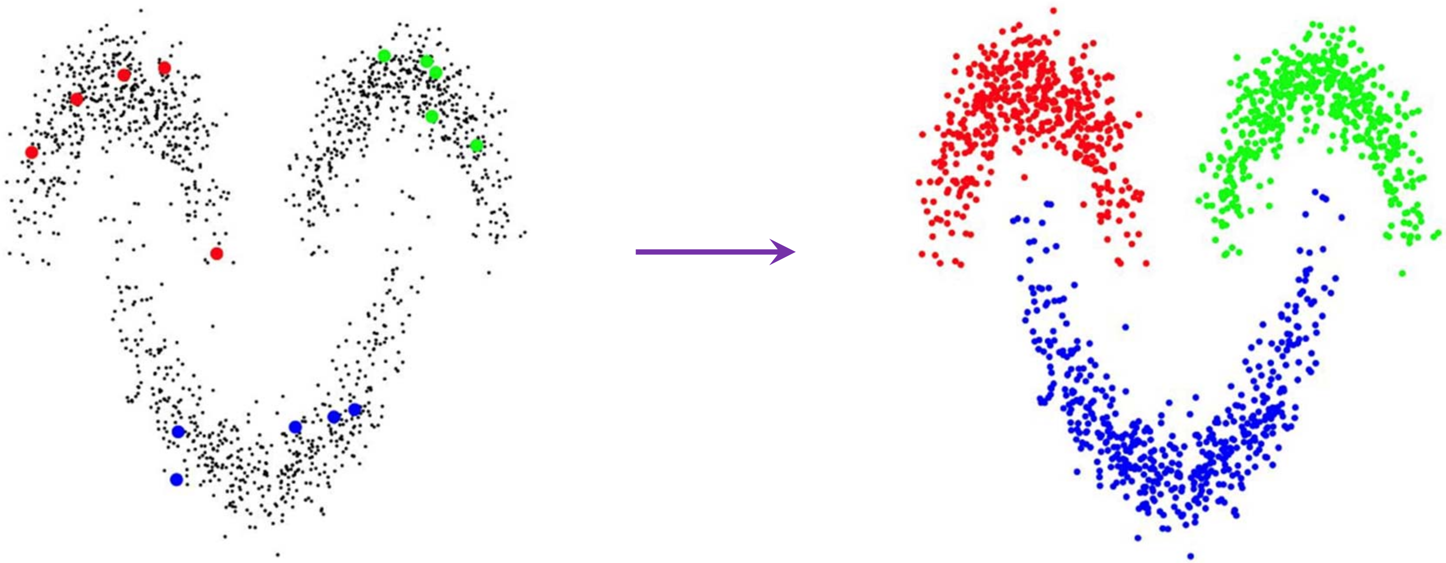
Accuracy over 10 random trials

	SaT with $\|\nabla g\|^2$	SaT without $\|\nabla g\|^2$	gain
overall accuracy	98.83%	97.26%	1.57%
average accuracy	98.88%	95.89%	2.99%
kappa	98.66%	96.86%	1.80%

- ☐ overall accuracy: percentage of correctly classified pixels
- ☐ average accuracy: average of the accuracy in each class
- ☐ kappa: Cohen's kappa coefficient

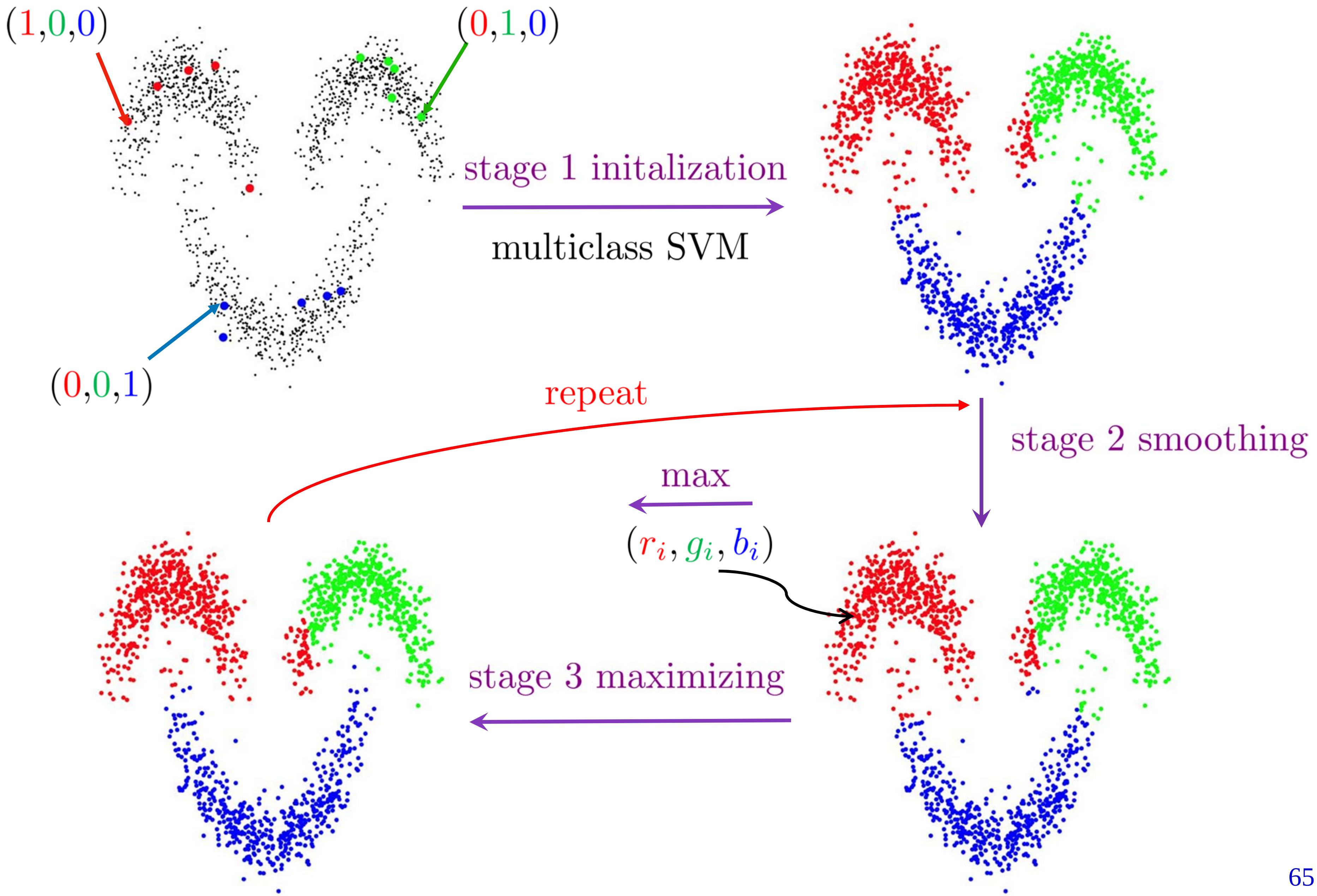
C., Kan, Nikolova, and Plemmons, arXiv 1806.00836.

Point Cloud Segmentation



10% training

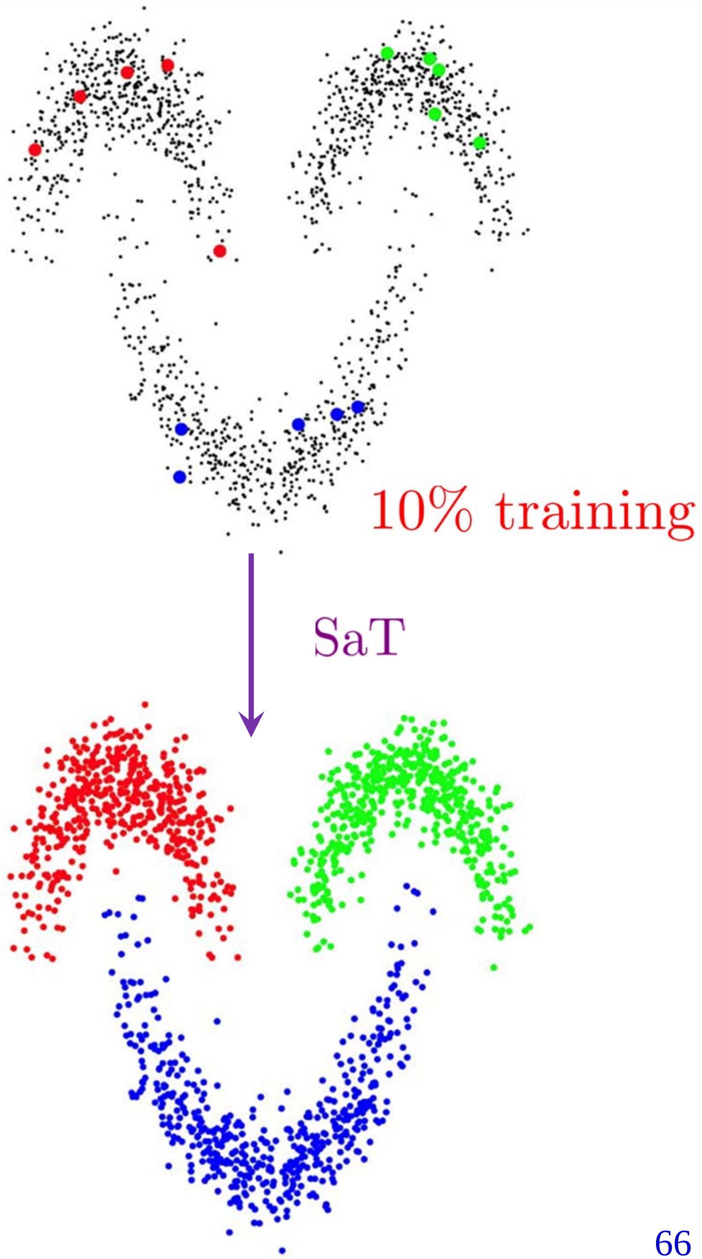
Our Smooth-and-Threshold (SaT) *Approach*



Result on 3-moon Segmentation

Average accuracy over 10 random trials

	accuracy
MBO [2014]	99.12%
GL [2014]	98.4%
Ke I [2016]	98.4%
Ke II [2016]	98.6%
CVM [2017]	98.71%
SaT method	99.23%
gain	0.11%



MBO, GL [Garcia-Cardona *et al.*, 2014],
Ke I, II [Ke & Tai, 2016],
CAM [Bae & Merkurjev, 2017]

Cai, C., Xie & Zeng, in preparation.

Conclusions

- SaT (Smooth-and-Threshold) framework looks for smooth solutions before segmenting or classifying
- Convex segmentation model with unique solution —can be solved easily and fast
- Model solved only once—no need to solve the model again when threshold or number of phases changes
- Easily extendable to blurry images, non-Gaussian noise, image with information loss, color images, hyper-spectral images, and point cloud images
- Link image segmentation and image restoration

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Thank You!

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