Image Segmentation and Understanding:
A Challenge for Mathematicians

Christoph Schnörr

Heidelberg University

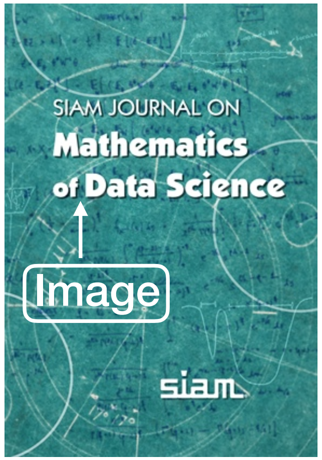
SIAM Conference on Imaging Science 2018
Bologna

Image Analysis: Mathematics and …

SIAM-IS'18

Inverse Scattering	Variational Approaches
Electr. Impedance Tomography	Regularization
Acousting Imaging	Optimization
Reconstruction from MRI	Diffusion
Imag. Micro- & Nano-structures	Denoising
Brain Imaging	Interpolation
Optical Coherence Tomography	Segmentation
Microscopic Imaging	Registration

SIAM News May 2018



J. Jost (MPI Math. in the Sciences)

Mathematics as a Tool:
Tracing New Roles of Mathematics
in the Sciences, Springer 2017

"… the distinction between
pure and applied mathematics
is no longer useful …"

MS16 - Topological Image Analysis:
Methods, Algorithms, Applications (3 parts)

algebraic topology
↓
persistent homology
↓
data analysis

Image Data Science

Computer Vision ?!	Math ?!!
Inverse Scattering	Variational Approaches
Electr. Impedance Tomography	Regularization
Acousting Imaging	Optimization
Reconstruction from MRI	Diffusion
Imag. Micro- & Nano-structures	Denoising
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Optical Coherence Tomography	Segmentation
Microscopic Imaging	Registration

Mathematics & (Computer) Vision

David Mumford Brown and Harvard Universities

What is "vision"?
...

What are the key mathematical tools appropriate for modeling the brain and cognitive skills?
...

Unwinding all these intertwined factors in order to infer the nature of world around you has proved to be very challenging and, as of this writing, is **only partially solved**.

Optimal Approximations by Piecewise Smooth Functions and Associated Variational Problems[1]

DAVID MUMFORD
Harvard University

AND

JAYANT SHAH
Northeastern University

1989

image partitioning
occlusion & scale-invariant models
nonlinear patch statistics
surface evolution & curvature flows
2D shape & conformal mapping
stochastic grammar of images

Grenander's pattern theory (Appl. Math. Brown University)
Geman, Bienenstock, Srivastava, Zhu, …

Image Data Science

Computer Vision

40 years (brief review): **4 approaches** ('paradigms')

framework containing the basic assumptions, ways of thinking, and methodology that are commonly accepted by members of the scientific community

dogma 

prescribed doctrine proclaimed as unquestionably true by the scientific community

Approach 1

"Reformulate CV as an *inverse problem*!" 1984

Early Vision: From Computational Structure to Algorithms and Parallel Hardware

TOMASO POGGIO

Artificial Intelligence Laboratory and Center for Biological Information Processing, Massachusetts Institute of Technology, 545 Technology Square, Cambridge, Massachusetts 02139

Received November 15 1984

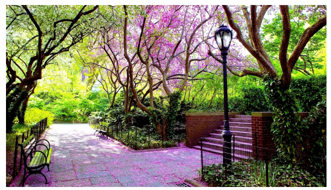
common computational structure of many early vision problems is that they are mathematically ill-posed in the sense of Hadamard. Standard regularization analysis can be used to solve

Approach 1

"Reformulate CV as an *inverse problem*!" 1984

≈1995 inverse optics approach: progress has remained elusive!

… unwinding all these intertwined factors …



Approach 2

"Active Vision Systems make CV well-posed" 1995

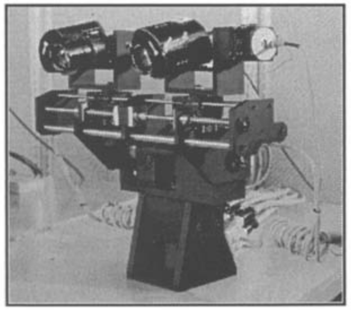
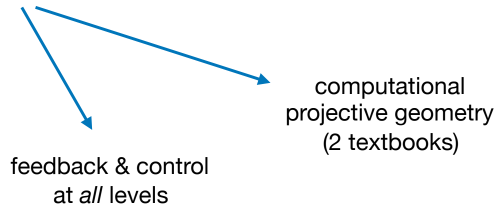


Figure 7. The KTH Head-Eye system was used for performing the experiments. The head-eye system consists of two cameras mounted on a neck and has a total of 13 degrees of freedom. It allows for computer-controlled positioning, zoom and focus of both the cameras independently of each other.

Approach 2

"Active Vision Systems make CV well-posed" 1995



Approach 3

"Bayesian Approach (Graph. Models, MRFs, CRFs)" 1985 - 2015

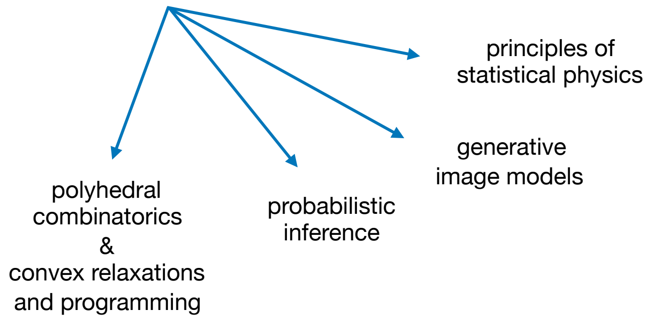
D. Mumford (pers. homepage):

> Below, a photo of my inspiration, Ulf Grenander, at his summer house in Sweden. He was the first to understand that Bayesian inference and graphical models were the best mathematical tools with which to model virtually all cognitive processes.

[PDF] Stochastic Relaxation, Gibbs Distributions, and the ... - Semantic Sch...
https://pdfs.semanticscholar.org/62c3/4c8a8d8b82a9c466c35cda5e4837c17d9ccb.pdf
by S GEMAN - Cited by 21625 Related articles
Oct 7, 1983 - STUART **GEMAN** AND DONALD **GEMAN**. Abstract-We make an analogy between images and statistical me- chanics systems. Pixel gray levels ...

Approach 3

"Bayesian Approach (Graph. Models, MRFs, CRFs)" 1985 - 2015



Approach 3

"Bayesian Approach (Graph. Models, MRFs, CRFs)" 1985 - 2015

$$\frac{1}{Z(\theta)} e^{\langle \theta, \phi(x) \rangle} \; ?$$

Approach 4

"Train a Deep Network !" 2012 - ????

unprecedented performance in many application areas

strong impact on large-scale optimisation (SGD, etc.)

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Vol. 60, No. 2, pp. 223–311

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Optimization Methods for Large-Scale Machine Learning*

Léon Bottou†
Frank E. Curtis‡
Jorge Nocedal§

- function: ~1000 image categories
- parameters: ~60.000.000
- training data: millions of labeled images

Approach 4

"Train a Deep Network !" 2012 - ????

"… understanding the reasons for this success has remained elusive."
"… the assumptions encoded in such models are for the most part a mystery."

(⟷ SIAM-IS'18: MS50, MS70)

arXiv: 2017 Failures of Gradient-Based Deep Learning

Shai Shalev-Shwartz[1], Ohad Shamir[2], and Shaked Shammah[1]

[1]School of Computer Science and Engineering, The Hebrew University
[2]Weizmann Institute of Science

training phase: adaptivity, feedback
test phase: black box

Image Segmentation and Understanding: A Challenge for Mathematicians

Christoph Schnörr

Ruben Hühnerbein, Fabrizio Savarino,
Alexander Zeilmann, Artjom Zern, Matthias Zisler

Freddie Aström

Heidelberg University

SIAM Conference on Imaging Science 2018
Bologna

Outline

Approach — Guiding Principle

1985

PDEs Variational

MS31-2

supervised labeling
assignment flow

smooth dynamical system

$$\dot{W}(t) = \mathcal{V}(W; D, G)$$

JMIV'17
SIIMS'18

1995

Action & Perception

MS31-1

unsupervised labeling
***coupled* assignment flow**

label *evolution*

$$(\dot{W}(t), \dot{G}(t)) = \mathcal{V}(W, G)$$

submitted

2005

Graphical Models

regularised labeling
***controlled* assignment flow**

learning by *control*

$$\dot{W}(t) = \mathcal{V}(W, U; D, G)$$

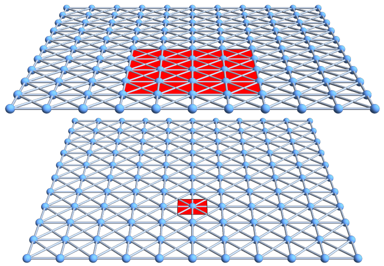
current

2015

Scope: Interplay of 2 levels

label space
(sign. statistics)

feature space
(high dim.)



patch

pixel

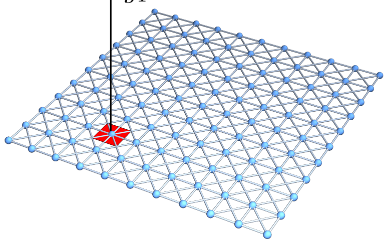
Supervised Labeling (Segmentation, Partitioning)

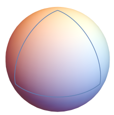
$D_i = \big(d(f_i, g_1), \dots, d(f_i, g_m)\big)$ metric, distance vector (**data term**)

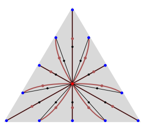
$W_i = \big(\Pr(g_1|f_i), \dots, \Pr(g_m|f_i)\big)$

g_m ⋮ g_2 g_1 labels (prototypes)

feature vector f_i







regularisation ?

$W_i \in (\mathcal{W}, g_{FR})$ assignment manifold

$f_i \in (\mathcal{X}, d)$ **features metric space**

Supervised Labeling (Segmentation, Partitioning)

$D_i = \big(d(f_i, g_1), \dots, d(f_i, g_m)\big)$ metric, distance vector (**data term**)

$\forall X, Y, Z \in \mathfrak{X}(\mathcal{W}):$

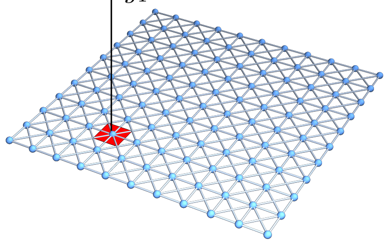
$Z g_{FR}(X, Y) = g_{FR}(\nabla_Z X, Y) + g_{FR}(X, \nabla^*_X Y)$

g_m ⋮ g_2 g_1 labels (prototypes)

feature vector f_i

Amari & Chentsov

$\overline{\nabla} = \frac{1}{2}(\nabla + \nabla^*)$



$W_i \in (\mathcal{W}, g_{FR})$ assignment manifold

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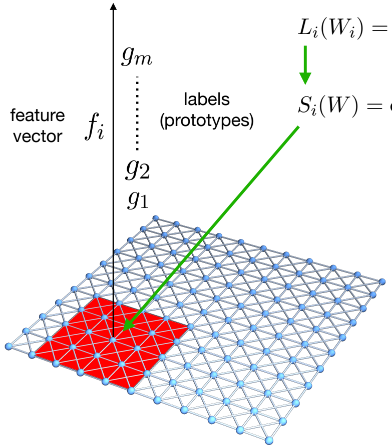
$L_i(W_i) = \exp_{W_i}\Big(-\frac{1}{\rho}\Pi_{W_i}(D_i)\Big)$ likelihood vector

$S_i(W) = \exp_{W_i}\Big(\sum_{j \in \mathcal{N}_i} w_j \exp^{-1}_{W_i}\big(L_i(W_i)\big)\Big)$ similarity vector



$W_i \in (\mathcal{W}, g_{FR})$ assignment manifold

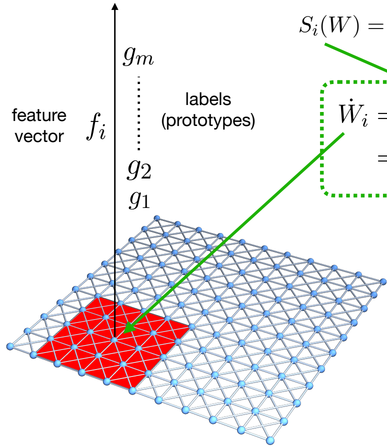
$f_i \in (\mathcal{X}, d)$ **features metric space**

Supervised Labeling (Segmentation, Partitioning) JMIV'17

$D_i = \big(d(f_i, g_1), \dots, d(f_i, g_m)\big)$ metric, distance vector (**data term**)

$S_i(W) = \exp_{W_i}\Big(\sum_{j \in \mathcal{N}_i} w_j \exp^{-1}_{W_i}\big(L_i(W_i)\big)\Big)$

$$\dot{W}_i = \Pi_{W_i}\big(S_i(W)\big) = W_i\big(S_i(W) - \langle W_i, S_i(W)\rangle \mathbb{1}\big)$$ assignment flow

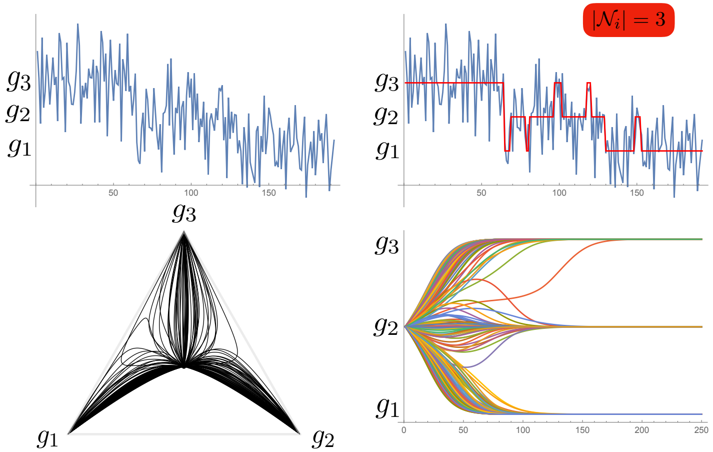


$W_i \in (\mathcal{W}, g_{FR})$ assignment manifold

$f_i \in (\mathcal{X}, d)$ **features metric space**

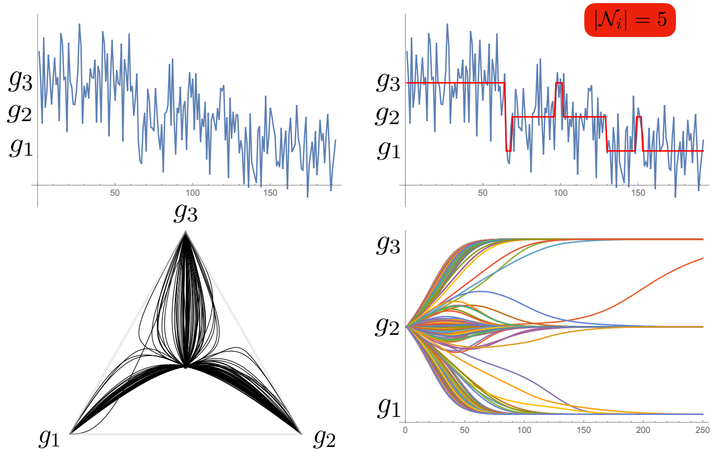
Supervised Labeling (Segmentation, Partitioning)

$|\mathcal{N}_i| = 3$

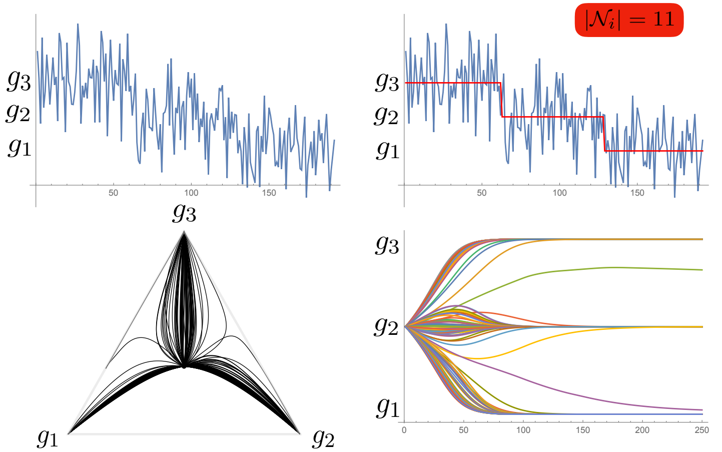


Supervised Labeling (Segmentation, Partitioning)

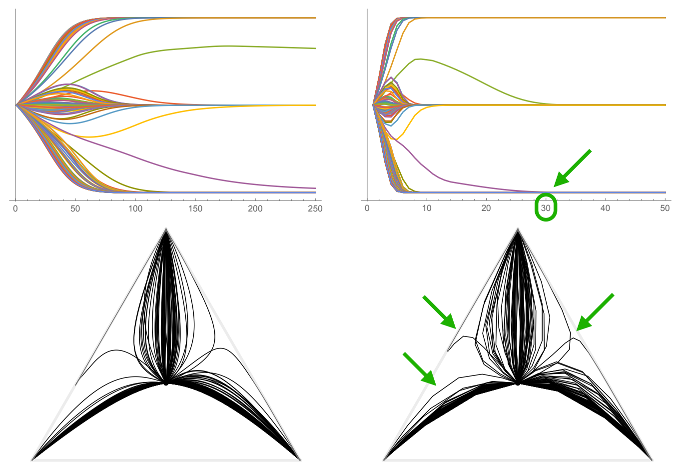
$|\mathcal{N}_i| = 5$



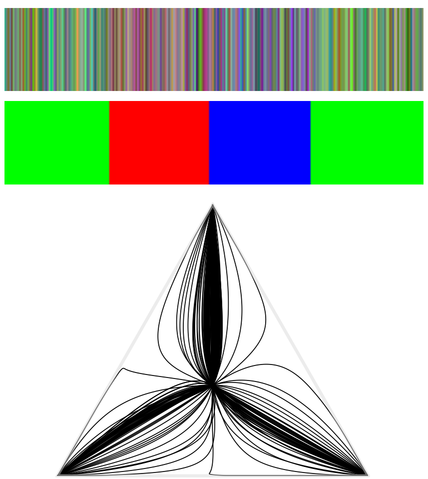
Supervised Labeling (Segmentation, Partitioning)



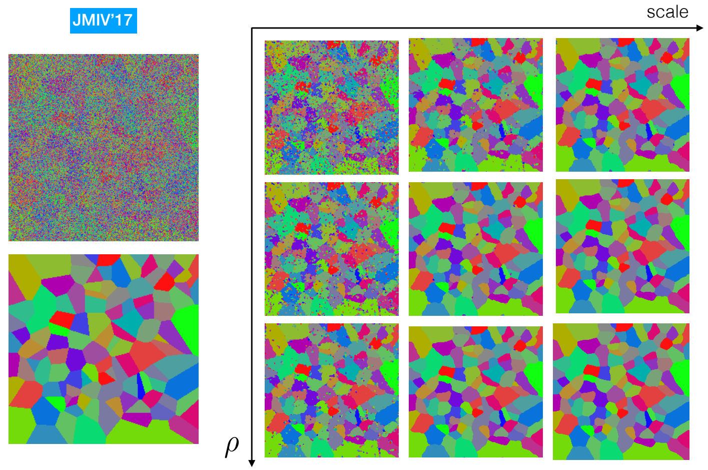
Supervised Labeling (Segmentation, Partitioning)



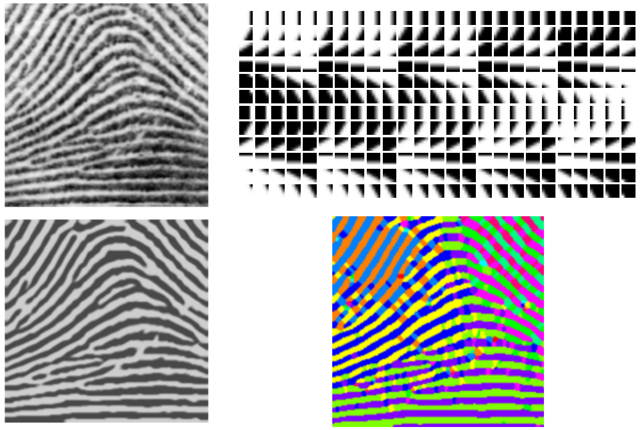
Supervised Labeling (Segmentation, Partitioning)



Supervised Labeling (Segmentation, Partitioning)



Supervised Labeling (Segmentation, Partitioning) JMIV'17



Supervised Labeling: Inference with Graphical Models SIIMS'18

$$E(x) = \sum_{i \in \mathcal{V}} E_i(x_i) + \sum_{ij \in \mathcal{E}} E_{ij}(x_i, x_j) \;, \qquad x_i \in \{g_1, \dots, g_m\}$$

optimal transport

$$E_{\tau}(W) = \sum_{i \in \mathcal{V}} E_i(W_i) + \sum_{ij \in \mathcal{E}} d_{ij\tau}(W_i, W_j)$$

"Wasserstein messages"

$$-\,\partial E_{\tau}(W)$$

smooth approximation

$$\dot{W} = \Pi_W\big(S(W)\big)$$

assignment flow

→ MS31-2

Supervised Labeling: Inference with Graphical Models SIIMS'18

"frustrated cycle experiment"

α	τ	Success rate	Iterations
0.22	0.2	97.35%	45
0.5	0.33	93.41%	15
0.58	0.15	88.6%	9

SIAM J. CONTROL AND OPTIMIZATION
Vol. 24, No. 5, September 1986

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008

DIFFUSIONS FOR GLOBAL OPTIMIZATION*

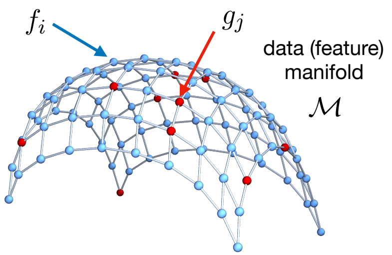
STUART GEMAN† AND CHII-RUEY HWANG‡

$$dx_t = -\nabla U(x_t)\, dt + \sqrt{2T}\, dw_t$$

Outline

Approach	Guiding Principle
MS31-2 *supervised* labeling **assignment flow**	*smooth dynamical* system
$\dot{W}(t) = \mathcal{V}(W; D, G)$	JMIV'17 SIIMS'18
MS31-1 *unsupervised* labeling ***coupled* assignment flow**	label *evolution*
$(\dot{W}(t), \dot{G}(t)) = \mathcal{V}(W, G)$	submitted
regularised labeling ***controlled* assignment flow**	*learning* by *control*
$\dot{W}(t) = \mathcal{V}(W, U; D, G)$	current

Unsupervised Labeling (Segmentation, Partitioning)



Subbarao & Meer (IJCV'09):
Nonlinear Mean Shift over Riemannian Manifolds

$$g_j^{(k+1)} = \exp_{g_j^{(k)}}\Big(\sum_i p_{ij}(G^{(k)})\widehat{g}^{-1}\big(d_j D(g_j^{(k)}, f_i)\big)\Big),\ j \in J$$

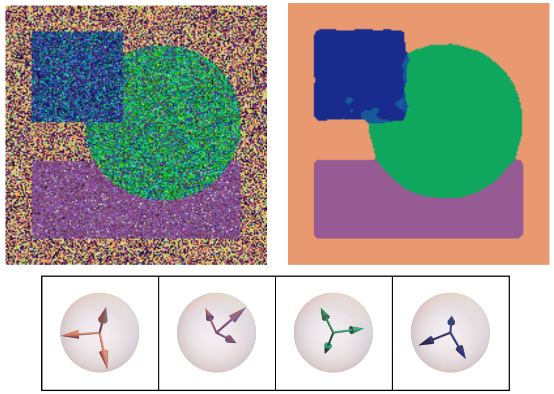
$$\big(\dot{W}(t), \dot{G}(t)\big) = \mathcal{V}(W, G)$$

coupled assignment flow:
spatially regularised label evolution

MS31-1

Unsupervised Labeling: Plug In and Play

SO(3)-valued data



Unsupervised Labeling: Plug In and Play

Euclidean color space

supervised: 200 labels

unsupervised: *few* labels

Unsupervised Labeling: Plug In and Play

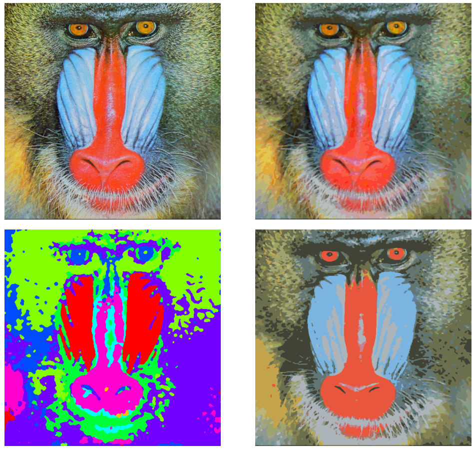
positive-def. manifold (dim = 120)

supervised: 200 labels

$$\mathcal{P}_d \ni F_i = \int h(x_i - y)\big[(f - \mathbb{E}_i[f]) \otimes (f - \mathbb{E}_i[f])\big](y)\,\mathrm{d}y$$

Unsupervised Labeling: Plug In and Play → MS31-1

supervised: 200 labels



few labels

few labels

Outline

Approach	Guiding Principle
supervised labeling **assignment flow** (MS31-2) $\dot W(t) = \mathcal{V}(W; D, G)$	*smooth dynamical* system — JMIV'17 SIIMS'18
<u>un</u>*supervised* labeling ***coupled* assignment flow** (MS31-1) $(\dot W(t), \dot G(t)) = \mathcal{V}(W, G)$	label *evolution* — submitted
regularised labeling ***controlled* assignment flow** $\dot W(t) = \mathcal{V}(W, U; D, G)$	*learning* by *control* — current

Feedback & Control - 1

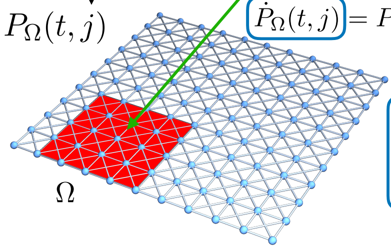
$$L_i(W_i) = \exp_{W_i}\Big(-\frac{1}{\rho}\Pi_{W_i}(D_i)\Big)$$ likelihood vector

$$S_i(W) = \exp_{W_i}\Big(\sum_{j\in\mathcal{N}_i} \boxed{w_j} \exp^{-1}_{W_i}\big(L_i(W_i)\big)\Big)$$ similarity vector

assigned label

$P_\Omega(t, j)$

$$\boxed{\dot P_\Omega(t, j)} = P_\Omega(t, j)\big(A_\Omega(t, j) - \mathbb{E}^t[A_\Omega(t)]\big)$$

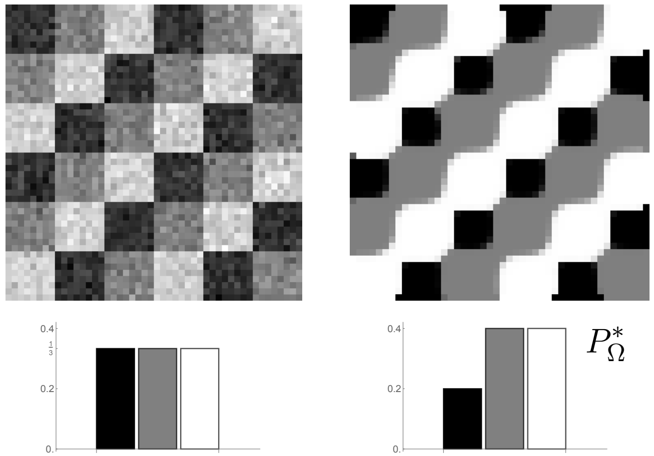


Ω

P^*_Ω **prescribed label statistics**

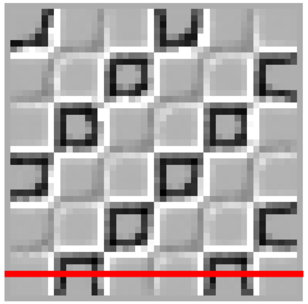
$D_{KL}\big(P^*_\Omega, P_\Omega(t)\big)$

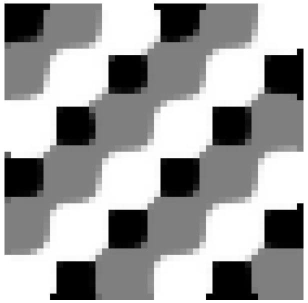
Feedback & Control - 1

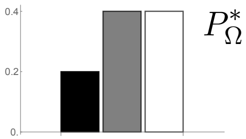


P^*_Ω

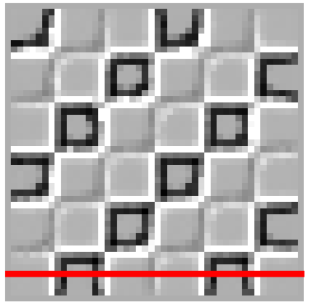
Feedback & Control - 1

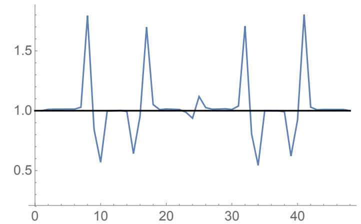


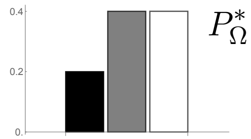




Feedback & Control - 1







Feedback & Control - 2

$$L_i(W_i) = \exp_{W_i}\Big(-\frac{1}{\rho}\Pi_{W_i}(\underbrace{D_i}_{\text{data}})\Big)$$

***decoupled* single pixel flow**

$$\dot{W}_i(t) = \Pi_{W_i(t)}\Big(L_i(W_i(t))\Big)$$

***short-time* single pixel flow**

$$W_i(t) = \exp_{W_{0,i}}\big(V_i(t)\big), \qquad W_{0,i} = W_i(t_0)$$

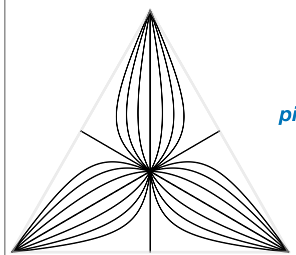
$$\dot{V}_i(t) = \Pi_{W_{0,i}}\Big(s + S\exp^{-1}_{W_{0,i}}\big(W_i(t)\big)\Big), \qquad V_i(t_0) = 0$$

***closed-form* single pixel flow ($t_0 \leftarrow 0$)**

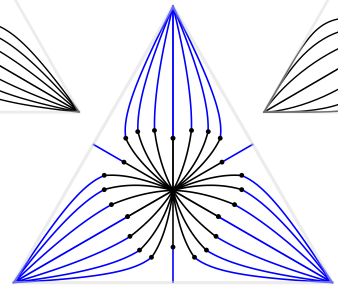
$$V_i(t) = \int_0^t e^{(t-\tau)\Pi_{W_{0,i}}(S)}\Pi_{W_{0,i}}(s)d\tau$$

Feedback & Control - 2

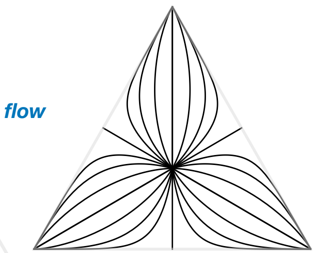
***full* single pixel flow**



piecewise linear flow



linearised flow



Feedback & Control - 2

short-time **full** pixel flow

$$W(t) = \exp_{W_0}\big(V(t)\big)$$

$$\dot{V}(t) = \Pi_{W_0}\Big(\underbrace{s + S\exp_{W_0}^{-1}\big(W(t)\big)}_{\textit{decoupled}} + U\big(t, W(t); D\big)\Big)$$

non-local coupling

On the mathematics of emergence⋆

Felipe Cucker · Steve Smale⋆⋆

Received: 17 October 2006 / Accepted: 24 January 2007

$$x(t+h) = x(t) + hv(t)$$

$$v_i(t+h) - v_i(t) = h\sum_{j=1}^{k} a_{ij}(v_j(t) - v_i(t)) \qquad a_{ij} = \frac{H}{(1+\|x_i - x_j\|^2)^{\beta}}$$

Related Publications

R. Hühnerbein, F. Savarino, F. Åström, and C. Schnörr. Image Labeling Based on Graphical Models Using Wasserstein Messages and Geometric Assignment. *SIAM J. Imaging Science*, 11(2):1317–1362, 2018.

F. Åström, S. Petra, B. Schmitzer, and C. Schnörr. Image Labeling by Assignment. *J. Math. Imag. Vision*, 58(2):211–238, 2017.

J.H. Kappes, B. Andres, F.A. Hamprecht, C. Schnörr, S. Nowozin, D. Batra, S. Kim, B.X. Kausler, T. Kröger, J. Lellmann, N. Komodakis, B. Savchynskyy, and C. Rother. A Comparative Study of Modern Inference Techniques for Structured Discrete Energy Minimization Problems. *Int. J. Comp. Vision*, 115(2):155–184, 2015.

F. Savarino, R. Garske, F. Åström, J. Recknagel, and C. Schnörr. Numerical Integration of Riemannian Gradient Flows for Image Labeling. In *Proc. SSVM*, LNCS 10302. Springer, 2017.

A. Zern, K. Rohr, and C. Schnörr. Geometric image labeling with global convex labeling constraints. In *EMMCVPR*, volume 10746 of *LNCS*, pages 533–547, 2018.